

Magneto-optic effects of the Cosmic Microwave Background

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Abstract

Generation of magneto-optic effects by the cosmic microwave background (CMB) in the presence of cosmic magnetic fields is studied. Four mechanisms which generate polarization of the CMB such as the Cotton-Mouton effect, the vacuum polarization in external magnetic field, the photon-pseudoscalar mixing in external magnetic field and the Faraday effect are studied. Considering the CMB linearly polarized at decoupling time due to Thomson scattering, it is shown that second order effects in the magnetic field amplitude such as the Cotton-Mouton effect in plasma and the vacuum polarization (Euler-Heisenberg term) in cosmic magnetic field, would generate elliptic polarization of the CMB at post decoupling time depending on the photon frequency and magnetic field strength. Among standard magneto-optic effects, the Cotton-Mouton effect in plasma turns out to be the dominant effect in the generation of CMB elliptic polarization in the low frequency part while the vacuum polarization in magnetic field is the dominant process in the high frequency part. The effect of pseudoscalar particles on the CMB polarization is also studied. It is shown that photon-pseudoscalar particle mixing in cosmic magnetic field generates elliptic polarization of the CMB as well, depending on the circumstances and even in the case of initially unpolarized CMB. New limits/constraints on the pseudoscalar parameter space are found. Prior decoupling CMB polarization due to pseudoscalar particles is also discussed.

1 Introduction

The interaction of light with matter and fields has been intensively studied in the literature and first quantitative studies dates back to Galileo, Newton, Faraday and Maxwell. Among the interesting effects that such interaction represents, there is one class of phenomena which includes the interaction of light (electromagnetic wave) with external electromagnetic fields. These phenomena manifest when an electromagnetic wave propagates through an external electromagnetic field that has been altered by the presence of the incident electromagnetic wave. If there is present only an external electric field, the effects that manifest are called electro-optic effects. Instead, if there is present only an external magnetic field, the effects that manifest belong to the category of the magneto-optic effects. In this work I study only the last effects.

Magneto-optic effects not only are important to the established physics but also allow to investigate new effects that have not been found yet. They are generally divided in three main categories that are related to transmission, reflection and absorption of the incident light by the magnetized medium. Depending on the initial polarization of the electromagnetic wave, there are essentially four magnetic-optic effects which belong to the transmission (and not only) category, the Cotton-Mouton (CM) effect, the Faraday effect and two more exotic effects which are the vacuum polarization and the mixing of photons with pseudoscalar (and also scalar) particles in external magnetic field. The reflection category includes essentially only the Kerr effect while the absorption category includes the so called molecular

circular dichroism in gases and as will be shown in this work also the photon-pseudoscalar mixing in magnetic field.

In the transmission category, the CM effect has been extensively studied in the literature. It has been experimented mostly in gases, liquids, solids and to some degree even in plasma. The CM effect manifest when light propagates in a magnetized medium where the external magnetic field has a transversal component with respect to the direction of light propagation [1]. This effect also shares a property with more exotic phenomena such as vacuum polarization (or simply QED effect) and photon-pseudoscalar mixing in magnetic field. All three effects manifest only where there is a transversal component of the external magnetic field with respect to the direction of light propagation.

The vacuum polarization has been first proposed and studied in Ref. [2] and since then has received much attention from both theory and experimental physics. This effect would manifest as a phase shift between the two photon states perpendicular and parallel to the external (transverse) magnetic field that eventually give rise to a birefringence effect which intensity depends on the incident electromagnetic wave frequency. One of the most important achievement from the experimental side, is to measure the acquired QED ellipticity angle of the incident light propagating through the magnetic field. Indeed, this has been the quest for the PVLAS [3] experiment, BFRT [4] experiment and for new generation of experiments [5]. After a first claim of detection of vacuum birefringence by PVLAS experiment [6], there is still a long way to achieve the required apparatus sensitivity in order to measure the QED predicted ellipticity which is by more than an order of magnitude smaller than the current apparatus sensitivity. At current status, apparatus sensitivity is contaminated with not well understood background noise, must probably from the same apparatus and new methods have also been proposed [7].

The birefringence effect predicted by QED can also be mimicked by another magneto-optical effect, namely the photon-pseudoscalar/scalar mixing in magnetic field. In fact, as it will be shown in this work, mixing of photons with pseudoscalar particles gives rise to both birefringence and dichroism effects. Therefore experiment such as PVLAS, BFRT etc., can in principle find pseudoscalar particles such as axions, ALPs, scalar bosons if the induced birefringence or dichroism signal is bigger than the QED expected signal. Other important experiments that aim to find exotic pseudoscalar particles include the CAST and IAXO experiments [8], ADMX experiment [9] and ALPS-II [10].

Among all magneto-optic effects, the Faraday effect has received much attention in astronomy and cosmology. It manifest when an initial linearly polarized electromagnetic wave interacts with an external magnetic field that has a longitudinal component along the wave propagation direction. This coupling makes possible the rotation of the polarization plane of the incident electromagnetic wave and the rotation angle is proportional to $B_e d$ where B_e is the strength of the external magnetic field and d is the length of the path. Consequently, the Faraday effect has been widely used in radio astronomy as a probe of cosmic magnetic fields, most probably with a primordial origin, in galaxy clusters and also in the intergalactic space [11]. Measurements of the rotation angle of light received from galaxy clusters confirm the presence of a magnetic field inside them, with a magnitude of about few μG . In the intergalactic space, present studies would suggest a weaker large scale magnetic field either homogeneous or stochastic with upper limit magnitude $B_e \lesssim 3 - 100 \text{ nG}$. On the other hand non observation of gamma rays emission from intergalactic medium due to injection of high energy particles by blazars would suggest a lower value on the strength of extragalactic magnetic field $B_e \geq 10^{-16} - 10^{-15} \text{ G}$ [12]. The origin of this field is still unknown and present studies suggest that it may have been created during structure formation or it may have a primordial origin, see Ref. [13] for a review on cosmic magnetic fields structure and current updated limits/constraints by Planck collaboration. In this work it is assumed that the magnetic field has a primordial origin with fixed direction in the sky and its amplitude is a slowly varying function of coordinates namely almost spatially homogeneous non stationary field.

In connection with the CMB physics, the Faraday effect has been used to probe the existence of primordial magnetic field [14] present at the decoupling time since it would rotate the polarization plane of the CMB. In fact, it is well known by now that the CMB posses a very small linear polarization that is believed to have been generated at the decoupling time due to Thomson scattering of CMB

photons on electrons. Such a polarization is generated because of temperature anisotropies present at the decoupling epoch that eventually generate a position dependent photon intensity on the surface of the last scattering [15]. Consequently, Thomson scattering of an anisotropic background of photons on electrons would generate linear polarization of the CMB with non zero Stokes parameters Q and U [16], [17].

In general, the linear polarization pattern of the CMB can be decomposed in two modes with opposite parity, the so called E-modes (or gradient modes G) which are the dominant component of the linear polarization and B-modes (or curl modes C) which are the subdominant component of linear polarization, see Refs. [18]. The former are generated only by scalar density fluctuations of the cosmological plasma while the latter are generated by either vector or tensor modes. The generation of B-modes is induced by tensor perturbations (gravitational waves) [19], Faraday rotation of the CMB [14], gravitational lensing of the E-mode component [20] and due to primordial magnetic fields [21] via perturbations sourced by the magnetic field. In general, the spectrum of B-modes is described in multipole moments l of spherical harmonics used to describe linear polarization. The location of the peak signal of B-modes as function of l would give the possibility to distinguish between signals generated by different sources of B-modes.

So far, much of attention on the CMB polarization has been focused mostly on the linear polarization. This fact, partially has been influenced by the experimental observation of E-modes (due to primordial adiabatic scalar fluctuations) by DASI, WMAP and BOOMERANG collaborations [22] and also by the fact that many inflationary models predict an almost scale invariant spectrum of gravitational waves, which as already mentioned above, can produce B-modes which are believed to be the ‘holy grail’ of the inflationary theory. Moreover, since Thomson scattering is the most frequent type of scattering in the early universe and because it generates only linear polarization, other types of CMB polarization have been to some extent obscured and the V Stokes parameter has become essentially the ‘lost along the way’ parameter. However, it is well known that light can have two additional types of polarization, circular and elliptic which translate into a nonzero Stokes parameter V .

After this premise on the CMB linear polarization, several questions come spontaneously. Does the CMB possess only linear polarization? Does it have any degree of circular polarization? If yes, what are the generating mechanisms? Even though, there is not urgency on the study of CMB circular polarization, since the discovery of the CMB, there have been several attempts in the past and also at the present to experimentally measure it. Moreover, since CMB linear polarization has already been detected by DASI, WMAP and BOOMERANG collaborations, the next step would be that of the study of circular polarization which as I will show in this paper is generated by very interesting mechanisms which are extremely important to the fundamental physics.

The first studies on the CMB circular polarization were done in connection with studies on anisotropic expansion of the universe which are characterized by some type of Bianchi models [23]. The first experimental attempts to measure the circular polarization of the CMB were done in Ref. [24] where no evidence for CMB circular polarization was found and only constraint on the degree of circular polarization was set. The current upper limit on the CMB circular polarization has been set by the MIPOL experiment [25], $P_C \lesssim 7 \times 10^{-5} - 5 \times 10^{-4}$ at the frequency 33 GHz and at angular scales between 8° and 24° .

In this work I study the impact of magneto-optic effects on the CMB polarization in the presence of cosmic magnetic fields. A systematical study of the most important magneto-optic effects in the generation of a net CMB elliptic (circular and linear) polarization is done. By including all magneto-optic effects mentioned above, I derive the equations of motion for the Stokes’s parameters which form a coupled system of differential equations. I use a density matrix approach to study the mixing of different magneto-optic effects and then solve the equations of motion by using perturbation theory. It turns out that among CM and vacuum polarization effects, the CM effect in plasma is the most promising effect in generation of elliptic polarization in the low frequency part of the CMB, while in the high frequency part, the vacuum polarization is the dominant one. I also will use current limit on the degree of circular polarization, to set new limits on the mass and coupling constant of pseudoscalar particles.

This paper is organized as follows: In Sec. 2, I derive the equations of motion for the photon and pseudoscalar fields in an expanding universe and introduce the photon polarization tensor in magnetized medium which describes forward scattering of photons. Moreover, I solve the equations of motion which describe the interaction of photon and pseudoscalar fields with the magnetized plasma in the case when the universe expansion can be neglected (for example laboratory conditions). Then I include the Faraday effect which makes possible the mixing of both photon states with the pseudoscalar field, which result in a 3×3 mixing matrix with two mixing angles. Also briefly discuss the connection of the photon density matrix with the Stokes parameter. In Sec. 3, I study the equations of motion for the density matrix in the case of open systems and establish the connection between the system Hamiltonian and the field mixing matrix. In Sec. 4, I find the equations of motion for the density matrix in an expanding universe and solve them in the case of vacuum polarization and CM effects. In Sec. 5, I present the equations of motion for the density matrix in the case when the contribution of the pseudoscalar field is included and introduce the concept of generalized Stokes parameters. Then I find perturbative solutions of the reduced Stokes vectors in transverse magnetic field. In Sec. 6, I study the generation of CMB circular polarization in the case of photon-pseudoscalar particle mixing in transverse magnetic field and set new limits on the pseudoscalar parameter space. In Sec. 7, I conclude. In this work I use the metric with signature $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ and work with the natural (rationalized) Lorentz-Heaviside units ($k_B = \hbar = c = \varepsilon_0 = \mu_0 = 1$) with $e^2 = 4\pi\alpha$.

2 Generalities

2.1 Equations of motions in an expanding universe

In this section we derive the equations of motion for the photon and pseudoscalar fields propagating in a magnetized medium in the framework of the Friedemann-Robertson-Walker (FRW) metric. To start with, we write the effective action of the photon and pseudoscalar fields in curved spacetime

$$\mathcal{S}_{\text{eff}} = \int d^4x \sqrt{-g} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \int d^4x' A_\mu(x) \Pi^{\mu\nu}(x, x') A_\nu(x') + \frac{1}{2} \partial_\mu \partial^\mu \phi - \frac{1}{2} m_\phi^2 \phi^2 + \frac{g_{\phi\gamma}}{4} \phi F_{\mu\nu} \tilde{F}^{\mu\nu} \right), \quad (2.1)$$

where $F_{\mu\nu}$ is the total electromagnetic field tensor, $\Pi^{\mu\nu}$ is the photon polarization tensor in medium, ϕ is the pseudoscalar field, m_ϕ is the mass of the pseudoscalar field, g is the metric determinant and A^μ is the photon vector potential. By varying the action with respect to the electromagnetic field A^ν and pseudoscalar field ϕ , the equations of motion are

$$\begin{aligned} \square A^\nu - \nabla_\mu (\nabla^\mu A^\nu) - \int d^4x' \Pi^{\mu\nu}(x, x') A_\mu(x') &= g_{\phi\gamma} (\partial_\mu \phi) \tilde{F}^{\mu\nu}, \\ (\square + m_\phi^2) \phi &= \frac{g_{\phi\gamma}}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}, \end{aligned} \quad (2.2)$$

where $\nabla_\mu \tilde{F}^{\mu\nu} = 0$, $\square = \nabla_\mu \nabla^\mu$ is the d'Alembertian operator in curved space, $x^\mu = (t, \mathbf{x})$, ∇_μ is the covariant derivative and $\nabla_\mu \phi = \partial_\mu \phi$. In this work we consider the case of flat ($\kappa = 0$) FRW metric with line element $ds^2 = dt^2 - d\mathbf{x}^2$, where t is the cosmological time and \mathbf{x} is the spatial coordinate. The only non zero components of the affine connection in the FRW metric are $\Gamma_{0j}^i = (\dot{a}/a) \delta_{ij}$ and $\Gamma_{ij}^0 = \dot{a} a \delta_{ij}$.

In general the electromagnetic field tensor $F_{\mu\nu}$ is given by the sum of the field tensor of incident photon field and of field tensor corresponding to the external magnetic field. In most cases the electromagnetic field tensor corresponding to the external magnetic field is the dominant term. Considering the photon propagation in external magnetic field, the equations of motion (2.2) for the components of vector

potential \mathbf{A}^i and pseudoscalar field ϕ in the Coulomb gauge¹ are

$$\begin{aligned} (\partial_t^2 - \nabla^2 + 3H \partial_t) \mathbf{A}^i + \int d^4 x' \Pi^{ij}(x, x') \mathbf{A}_j(x') &= -g_{\phi\gamma}(\partial_t \phi) \mathbf{B}_e^i, \\ (\partial_t^2 - \nabla^2 + 3H \partial_t + m_\phi^2) \phi &= g_{\phi\gamma} \partial_t \mathbf{A}_i \cdot \mathbf{B}_e^i. \end{aligned} \quad (2.3)$$

We may notice that there is an extra term in the equations of motion (2.3) with respect to the Minkowski flat space-time for the photon and pseudoscalar fields, that is $3H\partial_t$ where $H = \dot{a}/a$ is the Hubble parameter and $a(t)$ is the cosmological scale factor. This term is the so called Hubble friction that is responsible for the damping of the fields in an expanding universe.

We look for single wave vector solutions of Eqs. 2.3 of the form

$$\mathbf{A}_j(\mathbf{x}, t) = \sum_{\lambda} A_{\lambda}(\mathbf{k}, t) \mathbf{e}_j^{\lambda}(\hat{\mathbf{n}}) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad \phi(\mathbf{x}, t) = \phi(\mathbf{k}, t) e^{i\mathbf{k} \cdot \mathbf{x}}, \quad (2.4)$$

where \mathbf{k} is the photon wave vector, \mathbf{e}^{λ} is the photon polarization vector and λ is the photon polarization index. For simplicity, we consider an electromagnetic wave propagating along the observer's $\hat{\mathbf{z}}$ axis with $\mathbf{k} = (0, 0, k)$, $k = |\mathbf{k}|$. Without any loss of generality we choose the external magnetic field in the xz plane with coordinates $\mathbf{B}_e = (B_e \sin(\Phi), 0, B_e \cos(\Phi))$ where Φ is the angle between the magnetic field direction and photon wave vector \mathbf{k} , $\cos(\Phi) = \hat{\mathbf{B}}_e \cdot \hat{\mathbf{n}}$ with $\hat{\mathbf{n}} = \mathbf{k}/k$. Given the symmetry of the problem, only the transverse part of the external magnetic induces photon-pseudoscalar mixing. Inserting the expansion (2.4) into the equations of motion (2.3) we obtain

$$\begin{aligned} \left(i\partial_t - k + \frac{3}{2}iH \right) A_+(k, t) + M_+(k) A_+(k, t) + iM_F(k) A_{\times}(k, t) &= 0, \\ \left(i\partial_t - k + \frac{3}{2}iH \right) A_{\times}(k, t) + M_{\times}(k) A_{\times}(k, t) - iM_F(k) A_+(k, t) + iM_{\phi\gamma}(k) \phi(k, t) &= 0, \\ \left(i\partial_t - k + \frac{3}{2}iH \right) \phi(k, t) - iM_{\phi\gamma}(k) A_{\times}(k, t) + M_{\phi}(k) \phi(k, t) &= 0, \end{aligned} \quad (2.5)$$

where we used the WKB approximation, namely that $\partial_t |A_{\lambda}| \ll \omega |A_{\lambda}|$ and $\partial_t |\phi| \ll \omega |\phi|$. Indeed, this approximation is well satisfied since the variation in time of external potential (that is proportional to the external magnetic field amplitude) due to universe expansion is much smaller than photon/pseudoscalar frequency. We also used the fact that photons and pseudoscalar particles are assumed to be relativistic and expanded the operator $(\partial_t^2 - \nabla^2) \simeq 2k(-i\partial_t + k)$. The system of Eqs. 2.5 can be written in a matrix form as follows

$$\left(i\partial_t - k + \frac{3}{2}iH \right) \begin{pmatrix} A_+ \\ A_{\times} \\ \phi \end{pmatrix} \mathbf{I} + \begin{pmatrix} M_+ & iM_F & 0 \\ -iM_F & M_{\times} & iM_{\phi\gamma} \\ 0 & -iM_{\phi\gamma} & M_{\phi} \end{pmatrix} \begin{pmatrix} A_+ \\ A_{\times} \\ \phi \end{pmatrix} = 0. \quad (2.6)$$

Here A_+ is the photon state perpendicular to the transverse part of \mathbf{B}_e , A_{\times} is the photon state parallel to the transverse part of \mathbf{B}_e and \mathbf{I} the identity matrix. The diagonal elements of the mixing matrix M in Eqs. (2.6) are $M_+ = -\Pi^{22}/(2k)$, $M_{\times} = -\Pi^{11}/(2k)$ and $M_{\phi} = -m_{\phi}^2/(2k)$, while the off diagonal elements are $M_{\phi\gamma} = g_{\phi\gamma} B_e \sin(\Phi)/2$ and $iM_F = -\Pi^{12}/(2k)$ is the term that corresponds to the Faraday effect. The elements of the photon polarization tensor² $\Pi^{11}, \Pi^{22}, \Pi^{12}$ and Π^{21} are calculated in momentum space

¹In the Coulomb gauge there is also the equation of motion for A^0 (scalar potential) which is proportional to $(\nabla \cdot \phi) \mathbf{B}_e$ for a globally neutral medium. In this case the mixing problem has four coupled differential equations in the case when \mathbf{B}_e is not transverse. However, the effect of this equation to the mixing problem is very small, namely of the order $(g_{\phi\gamma} B_{eL})^2$ where B_{eL} is the magnitude of the longitudinal component of \mathbf{B}_e and therefore we can safely neglect this equation for our purposes [26].

²The elements of the photon polarization tensor in magnetized, non relativistic and non degenerate electron plasma calculated in Ref. [27], include only the Faraday effect and CM effect in plasma. They do not include the contribution of vacuum polarization in magnetic field and CM effect in gases.

[27] where we took the adiabatic limit $t' \rightarrow t$. Their expressions will be given explicitly in the next sections. As far as concerns the nature of the large scale magnetic field, in this work is assumed that it has fixed direction in the sky and its amplitude B_e is a slowly varying function of space coordinates, namely homogeneous or almost homogeneous in space. On the other hand due to universe expansion, the field amplitude changes in time, namely the large scale magnetic field is non stationary.

2.2 Field mixing in stationary background

In the previous section we found the equations of motion for the photon and pseudoscalar fields in the FRW metric and in WKB approximation. Analogous equations exist in Minkowski space-time [28] which are stationary equations for the photon/pseudoscalar propagation in transverse external magnetic field. They have been applied in many contexts including astrophysical and cosmological situations in order to calculate the transition probability of photons into pseudoscalar particles and vice-versa, in laboratory magnetic field and in galactic and intergalactic magnetic fields. However, in many astrophysical and cosmological scenarios, the direction of propagation of photons/pseudoscalar particles is not necessarily perpendicular to the external magnetic field but it may have an angle $\Phi \neq \pi/2$ if the field direction is known to be fixed and we observe at an angle $\Phi \neq \pi/2$ or one is interested in averaging over Φ if the field direction changes in time or space. In these cases, unavoidably it is induced the Faraday effect which makes possible the mixing of photon helicity states with each other and it is represented in the mixing matrix by the term iM_F .

In this section we solve Eqs. (2.6) in the case of time independent magnetic field where the universe expansion can be neglected in the case when only Faraday effect is present and the pseudoscalar field is absent and vice-versa. This situation is quite often in ground based experiments looking for axions and other similar particles and in cosmological situations of low redshift. The solutions presented in this section are important for two reasons: first we present a formalism for arbitrary Φ including the Faraday effect and second, it would give us a general idea which are the combined effects of the magnetized medium and pseudoscalar field on the photon polarization. Before attempting to solve Eqs. (2.6) it is important to slightly transform the mixing matrix M . As one can see, M has complex entries and is a hermitian one. In order to avoid of working with complex mixing angles and because our results would be more comprehensible, it is convenient to make the entries in M real. This can be easily done by a global phase transformation of the fields as follows: $A_+ \rightarrow iA_+$, $A_\times \rightarrow A_\times$ and $\phi \rightarrow -i\phi$. After this transformation and in the case when the elements of the mixing matrix M are independent of t , we find the following solutions³ for Eqs. (2.6) in case of missing pseudoscalar field

$$A_+(t) = \left[e^{i\lambda_1 t} \cos^2(\alpha) + \sin^2(\alpha) e^{i\lambda_2 t} \right] e^{-ikt} A_+(0) + \frac{1}{2} \sin(2\alpha) \left[e^{i\lambda_2 t} - e^{i\lambda_1 t} \right] e^{-ikt} A_\times(0), \quad (2.7)$$

$$A_\times(t) = \frac{1}{2} \sin(2\alpha) \left[e^{i\lambda_2 t} - e^{i\lambda_1 t} \right] e^{-ikt} A_+(0) + \left[\sin^2(\alpha) e^{i\lambda_1 t} + \cos^2(\alpha) e^{i\lambda_2 t} \right] e^{-ikt} A_\times(0), \quad (2.8)$$

where $t_0 = 0$ is the initial time which we choose to be zero for simplicity, α is the mixing angles of the A_+ and A_\times states and $\lambda_{1,2}$ are the eigenvalues of the mixing matrix M for missing pseudoscalar field. Their expressions are respectively given by

$$\tan(2\alpha) = \frac{2M_F}{M_\times - M_+}, \quad \lambda_{1,2} = \frac{M_+ + M_\times}{2} \pm \frac{M_+ - M_\times}{2 \cos(2\alpha)}.$$

In the case when the external magnetic field is transverse, namely $M_F = 0$, we get the following exact

³In what follows, for convenience reasons we will suppress the dependence of the fields on k .

solutions of Eqs. (2.6)

$$A_+(t) = e^{-ikt} A_+(0), \quad (2.9)$$

$$A_\times(t) = \left[e^{i\lambda_3 t} \cos^2(\beta) + e^{i\lambda_4 t} \sin^2(\beta) \right] e^{-ikt} A_\times(0) - \frac{1}{2} \sin(2\beta) \left(e^{i\lambda_3 t} - e^{i\lambda_4 t} \right) e^{-ikt} \phi(0), \quad (2.10)$$

$$\phi(t) = -\frac{1}{2} \sin(2\beta) \left(e^{i\lambda_3 t} - e^{i\lambda_4 t} \right) e^{-ikt} A_\times(0) + \left(\cos^2(\beta) e^{i\lambda_4 t} + \sin^2(\beta) e^{i\lambda_3 t} \right) e^{-ikt} \phi(0), \quad (2.11)$$

where β is the mixing angle of the states A_\times and ϕ in case of $M_F = 0$ and $\lambda_{3,4}$ are the eigenvalues of M for $M_F = 0$. Their expression are given by

$$\tan(2\beta) = \frac{2 M_{\phi\gamma}}{M_\phi - M_\times}, \quad \lambda_{3,4} = \frac{M_\times + M_\phi}{2} \pm \frac{M_\times - M_\phi}{2 \cos(2\beta)}$$

There are several important considerations that can be made about (2.7)-(2.11). In this section we are interested in the phase and magnitude change of the photon states A_+ and A_\times . To start with, we can see that in the case when the Faraday effect is absent (in the case of transversal magnetic field) and missing pseudoscalar field $M_\phi = 0, g_{\phi\gamma} = 0$, we have that time evolutions of the photon states are given by

$$A_+(t) = e^{i(M_+ - k)t} A_+(0), \quad A_\times(t) = e^{i(M_\times - k)t} A_\times(0),$$

and their relative phase shift due to the medium is $\delta_m(t) = (M_+ - M_\times)t$. This shows that in the absence of the Faraday effect and missing pseudoscalar field, an initially linearly polarized electromagnetic wave would develop an elliptic polarization if $M_+ \neq M_\times$, that is a magnetically birefringence effect. The most important mechanisms which we study and which generate extra phase shifts are the vacuum polarization in external magnetic field, the CM effect and photon-pseudoscalar particle mixing. Indeed, as we will see in more details in the following sections, for the first two processes the components of the photon polarization tensor (refraction indexes) of the states A_+ and A_\times are different from each other which imply non zero phase shift $\delta(t)$.

2.2.1 Weak and strong mixing with respect to the Faraday effect

In the case when only the Faraday effect is present, it would be more convenient first to separate our analysis in two cases with respect to the Faraday effect: weak and strong mixing. In the weak mixing case ($\alpha \ll 1$), the terms proportional to $A_+(0)$ and $A_\times(0)$ on the left hand side of (2.7) and (2.8) have real and imaginary parts that are different from zero. This statement can be verified as follows: first we write expressions (2.7)-(2.8) in the following form

$$\Psi(t) = \tilde{M} \Psi(0), \quad (2.12)$$

where $\Psi(t)$ is a two component field $\Psi(t) = (A_+, A_\times)^T$ where T is the usual transpose symbol of a vector and \tilde{M} is a complex matrix. It is convenient at this stage to measure the relative phase shift with respect to the state A_\times , namely we define up to a common phase $\Psi(t) \rightarrow e^{-i(M_\times - k)t} \Psi(t)$. Second, we are interested in the phase change of each photon state with respect to the initial state due to mixing with the other photon. Phases are in general encoded in the imaginary part of each photon state at a given time t , relative to the initial state at the time t_0 . Consequently from (2.7)-(2.8) and using (2.12), for $\alpha \ll 1$ we get

$$\begin{aligned} \text{Im}\{\tilde{M}_{11}\} &= \sin(\Delta M t) + \Delta M t \alpha^2 \cos(\Delta M t) - \alpha^2 \sin(\Delta M t) \\ \text{Im}\{\tilde{M}_{22}\} &= \alpha^2 [\sin(\Delta M t) - \Delta M t]. \end{aligned} \quad (2.13)$$

We may observe from (2.13) that there is a phase shift between the states A_+ and A_\times in the case when the Faraday effect is present and the photon states have $\Delta M \neq 0$. In general, the total induced phase

shift in the presence of the Faraday effect, vacuum polarization and CM effect is given (for very small phase shift) $\delta(t) \simeq \text{Im}\{\tilde{M}_{11}\} - \text{Im}\{\tilde{M}_{22}\}$. It is important to stress that Faraday effect induces additional phase shift only if $M_+ \neq M_\times$.

The matrix elements \tilde{M}_{11} and \tilde{M}_{22} have also real parts which are different from unity and are related to the magnitude of the states A_+ and A_\times respectively. Their expressions in the weak mixing case ($\alpha \ll 1$) are given by

$$\begin{aligned} \text{Re}\{\tilde{M}_{11}\} &= \cos(\Delta Mt) + \alpha^2[1 - \cos(\Delta Mt) - \Delta Mt \sin(\Delta Mt)], \\ \text{Re}\{\tilde{M}_{22}\} &= 1 - 2\alpha^2 \sin^2\left(\frac{\Delta Mt}{2}\right). \end{aligned} \quad (2.14)$$

We come to the conclusion that in the case of (initially) elliptical polarized light ($\Delta M \neq 0$) in the presence of a longitudinal magnetic field (Faraday effect), two effects would manifest in the weak mixing case: an extra phase shift with respect to the case of no Faraday effect and the magnitude of each photon state is reduced according to (2.14). The second effect would induce a rotation of the polarization plane of the incident light⁴. It is worth to note that in the weak mixing case $\Delta M \neq 0$.

In the strong mixing case ($\alpha = \pi/4$ or $\Delta M = 0$) we have that

$$\text{Im}\{\tilde{M}_{11}\} = \text{Im}\{\tilde{M}_{22}\} = 0. \quad (2.15)$$

As one can see from (2.15) both photon states do not acquire any additional phase shift due to the Faraday effect. Therefore contrary to the weak mixing case, there is not a total phase shift (because $\Delta M = 0$) and no induced birefringence effect. A linearly polarized photon would have its polarization state unchanged in time. This means that the Faraday effect does not affect linearly polarized light which is a well known classical effect. In the strong mixing case there are also real parts which are different from unity

$$\text{Re}\{\tilde{M}_{11}\} = \text{Re}\{\tilde{M}_{22}\} = \cos(M_F t)$$

where both real parts are equal for both photon states. In the strong mixing case with respect to the Faraday effect, the light remains linearly polarized if initially is and there is only a rotation of the polarization plane with no induced magnetic birefringence.

2.2.2 Weak and strong mixing with respect to the pseudoscalar field

Now we focus on the effect of the pseudoscalar field on the photon polarization. So far, we studied the weak and strong mixing cases only with respect to the Faraday effect. The weak and strong mixing cases with respect to the pseudoscalar field are analogous to the Faraday effect. The imprints of the pseudoscalar production on the photon polarization are best understood in the case of transverse magnetic field, that is a missing Faraday effect. As we did in the previous section we separate our analysis in two cases: weak and strong mixing with respect to β .

As we did in the previous section we can write solutions (2.10)-(2.11) in the following form $\Psi'(t) = \tilde{M}'\tilde{\Psi}'(0)$ where $\Psi'(t) = (A_\times, \phi)^T$ and define $\Psi'(t) \rightarrow e^{-i(M_\times - k)t}\Psi'(t)$. In the weak mixing case, the effect of pseudoscalar production on the photon polarization appears in the first term on the r. h. s. of (2.10) or in the matrix element \tilde{M}'_{11} since it is only the state A_\times which mixes with the pseudoscalar field in transverse magnetic field. The additional phase shift⁵ $\delta_\phi(t)$, induced by pseudoscalar production is encoded in the imaginary part of \tilde{M}'_{22} . Expanding \tilde{M}'_{22} in series for $\beta \ll 1$ we get

$$\delta_\phi(t) \simeq \text{Im}\{\tilde{M}'_{22}\} = \beta^2(\Delta M_2 t - \sin(\Delta M_2 t)), \quad \Delta M_2 \equiv M_\times - M_\phi.$$

⁴In this section we are not interested in the calculation of the rotation angle of the polarization plane or in the ellipticity angle. They can be easily calculated by using standard methods.

⁵There is still present the phase shift $\delta_m(t) = (M_+ - M_\times)t$ between A_+ and A_\times .

The extra phase induced by pseudoscalar production, obviously sum up with the phase induced by vacuum polarization and CM effects. On the other hand the real part of \tilde{M}'_{22} is directly connected with the magnitude of the state A_\times . Indeed, the real part of \tilde{M}'_{22} for $\beta \ll 1$ is given by

$$\text{Re}\{\tilde{M}'_{22}\} \simeq 1 - 2\beta^2 \sin^2(\Delta M_2 t/2),$$

and the magnitude of the state A_\times is reduced by a factor $\text{Re}\{\tilde{M}'_{22}\}$ with respect to its initial value $|A_\times(0)|$ for $\phi(0) = 0$. Therefore in the weak mixing case, we can see that pseudoscalar production gives rise of two effects: it induces an extra phase shift and reduces the magnitude of the mixed state A_\times .

In the strong mixing case ($\beta = \pi/4$ or $\Delta M_2 = 0$), from (2.8) we get

$$\text{Im}\{\tilde{M}'_{22}\} = 0, \quad \text{Re}\{\tilde{M}'_{22}\} = \cos(M_{\phi\gamma} t). \quad (2.16)$$

From (2.16) we can observe some differences with respect to the weak mixing case. First, there is not an induced additional phase shift between photon states A_\times and A_+ due to the pseudoscalar field (no induced pseudoscalar birefringence effect) and second, the magnitude of the mixed photon state is reduced by a different factor with respect to the weak mixing case. The change in magnitude of the mixed state A_\times would give rise to a rotation of the polarization plane of photons (incident electromagnetic wave).

In the weak mixing case, photons and pseudoscalar particles are in a mixed state and there is not a complete transition between the two particles. In the strong mixing case, pseudoscalar particles are no longer virtual particles and are resonantly produced (complete transition). In the latter case only a dichroism effect manifest without a magnetically birefringence effect, while in the former case both dichroism and magnetically birefringence are present, see Fig. 1. In the case when both Faraday effect and pseudoscalar field are present obviously there is a combination of both effects in generation of photon polarization where both dichroism and birefringence effects would be present⁶.

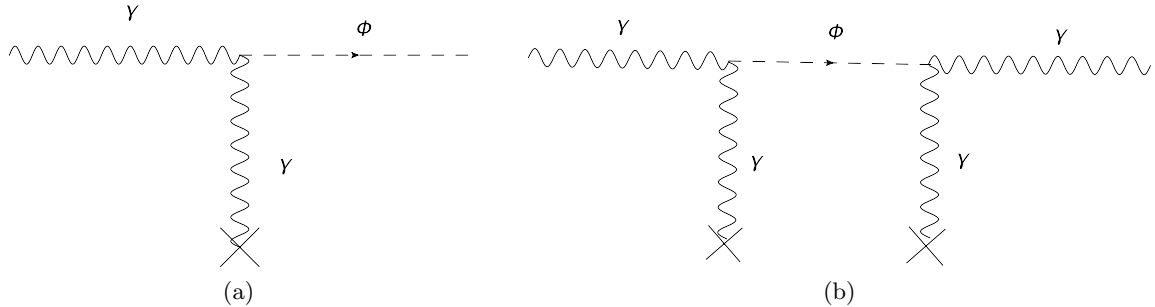


Figure 1: In (a) the complete transition of a photon into a pseudoscalar particle in external magnetic field (cross vertexes) is shown. Such a process occurs in the strong mixing case and leads to a dichroism effect. In (b) the pseudoscalar mediated photon-photon oscillations in external magnetic field is shown. This process leads to magnetically birefringence in external field.

2.3 Photon density operator and Stokes parameters

In most cases we have to deal with quantities that are proportional to the amplitude square of fields rather than the amplitude itself and that are connected with photon polarization. Such quantities are the Stokes parameters which give a complete description of photon polarization state. Here we briefly discuss their connection with photon polarization density matrix which we will extensively use in this paper.

⁶The combined effects of pseudoscalar field and Faraday effect on the photon polarization can be found by first solving Eqs. (2.6) exactly and second by proceeding in the same way as we did in this section in order to find generation of phase shifts and rotation of the polarization plane of incident light.

Use of Stokes parameters to describe photon polarization is very convenient because allow us to deduct important information about photon polarization by using four measurable quantities associated with the photon field. The first quantity or observable expresses the intensity of the photon field while the remaining three quantities completely describe its polarization state. Stokes parameters can be applied to unpolarized, partially polarized and completely polarized light and have the mathematical convenience of not being expressed in terms of the photon amplitude, which is in general not observable. However, an observable quantity is the photon field intensity which is derived by taking the time average of the square of the amplitude.

Consider a plane wave (not necessarily monochromatic) propagating along the z direction in a given cartesian coordinate system and consider the wave at $z = 0$. The wave electric field vector can be decomposed along the x and y components as follows $E_x(t) = E_{x0}(t) \cos[\omega t + \delta_x]$, $E_y(t) = E_{y0}(t) \cos[\omega t + \delta_y]$, where δ_x, δ_y are respectively the instantaneous wave phases for each field component, E_{x0} and E_{y0} are respectively the instantaneous wave amplitudes and ω is the instantaneous wave angular frequency. Here we work under the hypothesis that electric fields amplitudes E_{x0}, E_{y0} and field phases δ_x, δ_y slowly fluctuate in time in comparison with the rapid vibration of cosine functions. In case of nearly monochromatic wave, the Stoke's parameters are defined as follows

$$\begin{aligned} I &\equiv \langle E_{x0}^2(t) \rangle + \langle E_{y0}^2(t) \rangle, & Q &\equiv \langle E_{x0}^2(t) \rangle - \langle E_{y0}^2(t) \rangle, \\ U &\equiv \langle 2E_{x0}(t)E_{y0}(t) \cos \delta(t) \rangle, & V &\equiv \langle 2E_{x0}(t)E_{y0}(t) \sin \delta(t) \rangle, \end{aligned} \quad (2.17)$$

where the symbol $\langle (...) \rangle$ indicates a time average over several periods. The parameter I in (2.17) represents the intensity of the light, the parameter Q describes the amount of linear horizontal or linear vertical polarization, U describes the amount of linear polarization at an angle $\pm\pi/4$ with respect to the propagation direction and V describes the amount of left or right circular polarization.

The quantities Q and U in general depends on the orientation of the coordinate system used for measurements but the quantities $Q^2 + U^2$, I and V are invariant under such orientation. It is straightforward to show that the orientation angle ψ of the polarization ellipse can be expressed in terms of Q and U as

$$\tan(2\psi) = U/Q,$$

where $0 \leq \psi < \pi$ physically represents the polar angle of the polarization ellipse. One can also express the total degree of polarization⁷ of the wave in terms of the Stokes parameters as follows

$$P = \frac{\sqrt{Q^2 + U^2 + V^2}}{I},$$

where $0 \leq P \leq 1$. If $P = 1$ the wave is completely polarized, if $P = 0$ the wave is completely unpolarized (natural light, $Q = U = V = 0$) and if $P < 1$ the wave is partially polarized. The degree of linear polarization is given by $P_L = \sqrt{Q^2 + U^2}/I$ and the degree of circular polarization is given by $P_C = |V|/I$.

The description of polarized light in quantum optics is different from classical optics and in general their connection is not obvious. In classical optics, polarization is described in terms of amplitudes and polarization ellipse while in quantum optics it is described in terms of the density matrix. Having defined the Stoke's parameters in (2.17), it is desirable to connect them with the polarization density matrix of a quantum optical system. As shown in Ref. [29], Stokes parameters are very important tool for treating polarization problems in both quantum and classical optical systems. In order to outline their connection with the polarization density matrix, let $|A\rangle$ be an arbitrary photon state which is a linear superposition of the quantum polarization states $|A_+\rangle$ and $|A_\times\rangle$

$$|A\rangle = c_1|A_+\rangle + c_2|A_\times\rangle,$$

⁷In what follows the letter P should not be confused with oscillation probabilities of the previous section.

where c_1, c_2 are complex amplitudes. Their absolute value square represents the probability to find a photon respectively in the state $|A_+\rangle$ or $|A_\times\rangle$. In both classical and quantum optics, the polarization state of the wave is completely described in terms of complex amplitudes c_1, c_2 and one can define the elements of the density matrix ρ as

$$\rho_{ij} = c_i^* c_j, \quad (i, j = 1, 2).$$

If F is any observable of the system, its expectation value on an arbitrary state is given by $\langle F \rangle = \text{Tr}(F_{ij}\rho_{ij})$ where summation over repeated indices is used.

Following Ref. [29], one can associate to the Stokes parameters their corresponding quantum mechanical operators⁸ as

$$\begin{aligned} \hat{I} &= |A_+\rangle\langle A_+| + |A_\times\rangle\langle A_\times|, & \hat{U} &= |A_+\rangle\langle A_\times| + |A_\times\rangle\langle A_+|, \\ \hat{V} &= i(|A_\times\rangle\langle A_+| - |A_+\rangle\langle A_\times|), & \hat{Q} &= |A_+\rangle\langle A_+| - |A_\times\rangle\langle A_\times|, \end{aligned}$$

where the expectation value of each operator is given by

$$\begin{aligned} I &= \langle \hat{I} \rangle = \text{Tr}(\rho_{ij}I_{ij}) = \rho_{11} + \rho_{22}, & U &= \langle \hat{U} \rangle = \text{Tr}(\rho_{ij}U_{ij}) = \rho_{12} + \rho_{21}, \\ V &= \langle \hat{V} \rangle = \text{Tr}(\rho_{ij}V_{ij}) = i(\rho_{12} - \rho_{21}), & Q &= \langle \hat{Q} \rangle = \text{Tr}(\rho_{ij}Q_{ij}) = \rho_{11} - \rho_{22}. \end{aligned} \quad (2.18)$$

One can find after some trivial algebra the polarization density matrix in the basis spanned by the photon states $|A_+\rangle, |A_\times\rangle$

$$\rho = \frac{1}{2} \begin{pmatrix} I + Q & U - iV \\ U + iV & I - Q \end{pmatrix}. \quad (2.19)$$

Expression (2.19) is an important representation of the density matrix in terms of the Stokes parameters and is very useful in many contexts. It is important to note that representation (2.19) is in the basis of the vector potential states and not in the electric field basis which is the most common used case. Consequently, the physical dimensions of the density matrix (and also Stokes parameters) in the vector potential basis are different from those in the electric field basis.

3 Open systems

In this section we consider the case when photons (for example the CMB) are considered to interact with a medium, which for example can be magnetic field and cosmological plasma. Our goal is to find the equation of motion for the density matrix which in general is not trivial. As mentioned earlier, we are interested in quantities that are proportional to the amplitude square of the fields and because we want to study the mixing of CMB photons with pseudoscalar particles, the density matrix approach is the most adapted in this situation. Another fact in favour of this approach is that the CMB is almost unpolarized where the statistical mixture is maximal and the description of such a state demands the use of the density matrix. In the case when a system couples to another system, we are dealing with open systems that exchange energy and matter between each other. Therefore, in the case of photons interacting with plasma and magnetic field, the photon number is not in general conserved and the most important processes that can change their number, in the case that we treat in this work, is photon-pseudoscalar particle mixing in the cosmological plasma.

In the general case of an open quantum system, the equations of motion for the total density matrix, in the Schrödinger picture, are given by the von Neumann equation

$$i\frac{\partial \rho}{\partial t} = [H_T, \rho], \quad (3.1)$$

⁸Connection of Stokes parameters with polarization density matrix is also briefly discussed in Ref. [16].

where ρ is the total density matrix of the system and H_T is the total Hamiltonian (not to be confused with the Hubble parameter of the next section). The total system, is in general the sum of a quantum system S which is coupled to another quantum system B which is called the environment or bath, namely $S + B$. The total system considered here is assumed to be closed, following Hamiltonian dynamics. The state of the system S , which we call the photon-pseudoscalar system, will change as a consequence of its internal dynamics and because of the interaction with its surroundings. The interaction leads to system-environment correlations, such that state changes of S , can no longer be represented in terms of unitary Hamiltonian dynamics. In this context, the photon-pseudoscalar system S is also called a reduced system.

Suppose that \mathcal{H}_S is the Hilbert space of the photon-pseudoscalar system S and \mathcal{H}_B is the Hilbert space of the environment. The Hilbert space of the total system $S + B$ would be the tensor product $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_B$ and the total Hamiltonian has the general form $H_T = H_S \otimes I_B + I_S \otimes H_B + H_I(t)$, where H_S is the free Hamiltonian of the reduced system, H_B is the free Hamiltonian of the environment, H_I is the interaction Hamiltonian between the two systems S and B and I_B, I_S are identity operators in their corresponding Hilbert spaces. If we are interested in the observables of only system S , we can define the density operator of such system by taking the partial trace on the total density operator of the system ρ as follows

$$\rho_S = \text{Tr}_B[\rho], \quad (3.2)$$

where ρ_S is the density operator of the system S (photon-pseudoscalar system) and Tr_B is the partial trace over the environment degrees of freedom. Inserting (3.2) into the von-Neumann equation we get

$$i \frac{\partial \rho_S}{\partial t} = \text{Tr}_B[H_T, \rho]. \quad (3.3)$$

Equation (3.3) is a general result which describes the evolution in time of the reduced system interacting with an *arbitrary* medium. The explicit form of the expression $\text{Tr}_B[H_T, \rho]$ on the r. h. s. of Eq. (3.3), generally depends on different processes that appears in a specific problem and on type of fields that interact with the system. In our case we deal with photons that interact with different particle fields in the cosmological plasma, such as electrons, positrons, protons, light nuclei, cosmic magnetic field and in principle with other exotic particles. We refer to these fields as background fields and the calculation of the expression $\text{Tr}_B[H_T, \rho]$ would be quite involved. In fact, as one may realize at this point, there are essentially two ways on writing down the equations of motion for ρ_S . The first possibility would be to start from the general expression (3.3) and use the Hamiltonian of the total system and calculate the commutator with ρ by taking the partial trace over B . The second possibility would be to start with the effective action and derive the equations of motions for the fields by including the effective polarization tensor for photons and their interaction with the pseudoscalar field. In the latter case, one can derive a Schrödinger type equation, which dynamics is governed by an effective Hamiltonian that is given by the mixing matrix M and a ‘damping’ term due to the Hubble friction as in (2.6). Obviously, the second method is more convenient since it bypasses all the tedious procedure in calculating the r. h. s. of Eq. (3.3). Similar approach has been widely used also in neutrino physics [30] and it is still the most used approach on calculating oscillation probabilities in presence of damping. However, the second approach mentioned above is an approximation of the first method and should not be sought as the most standard procedure.

All told, we work under the approximation

$$\text{Tr}_B[H_T, \rho] \approx [M, \rho_S] - i\{D, \rho_S\}, \quad (3.4)$$

where M is the field mixing matrix that is already ‘traced out’ since it includes the effect of background fields on photons, photon-pseudoscalar interaction and D is a ‘damping’ matrix that is given by $D = (3/2)H\mathbf{I}$ where H is the Hubble parameter. On the right hand side of (3.4) instead of the total system density matrix appears only the reduced system density matrix ρ_S . This is due to the fact that the

coupling between S and B is weak such that the influence of S on B is very small (the so called Born approximation). In such case, at a given time t one can approximate⁹ $\rho(t) \approx \rho_S \otimes \rho_B$ [31]. Consequently, the equation of motion for the density matrix becomes

$$\frac{\partial \rho_S}{\partial t} = -i[M, \rho_S] - \{D, \rho_S\}, \quad (3.5)$$

where the first term in (3.5) describes an unitary evolution and the second term describes the ‘damping’ of fields in an expanding universe.

4 Photon polarization effects

In this section we focus on the case when the mixing matrix M is not stationary and look for solutions of equations of motion of the density matrix, Eq. (3.5). Indeed, it is more convenient to work with the density matrix than the wave equation, Eq. (2.6). In this section we consider the case of missing pseudoscalar field. As already mentioned, in the presence of an external magnetic field, excluding for the moment the case of photon-pseudoscalar mixing, there are essentially other three magneto-optic effects which depend on the external magnetic field direction and which are proportional to its strength.

In the presence of vacuum polarization, CM and Faraday effects the equation of motion of the density matrix are given by

$$\begin{aligned} \dot{I} &= -3HI, \\ \dot{Q} &= -2M_F U - 3HQ, \\ \dot{U} &= 2M_F Q + (M_+ - M_\times) V - 3HU, \\ \dot{V} &= -(M_+ - M_\times) U - 3HV \end{aligned} \quad (4.1)$$

where we wrote the elements of the density matrix in terms of the Stokes parameters. In general, the r. h. s. of (4.1) would depend on the temperature T rather than t . Therefore, expressing the time as $t = t(T)$, the time derivative in the FRW metric becomes $\partial_t = -HT\partial_T$ where $H = -\dot{T}/T$. The system (4.1) can be written in the following matrix form

$$S'(T) = A(T) \cdot S(T) + (3/T)\mathbf{I} \cdot S(T), \quad (4.2)$$

where S is the Stokes vector¹⁰ defined as $S = (I, Q, U, V)^T$ and $A(T)$ is a matrix defined as

$$A(T) = \frac{1}{HT} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2M_F(T) & 0 \\ 0 & -2M_F(T) & 0 & -\Delta M(T) \\ 0 & 0 & \Delta M(T) & 0 \end{pmatrix}.$$

The system (4.2) is a first order system of linear differential equations with variable coefficients. Even though the matrix $A(T)$ that enters (4.2) looks very simple, generally the system (4.2) has no closed form of solutions. However, it is possible to find analytic solutions by using the perturbation theory. Indeed, as we will see in what follows, for the parameter space of the photon/pseudoscalar momentum k and magnetic field strength B_e which we study in this work, one has in most cases the condition $M_F \gg |\Delta M|$. This condition on the other hand depends on Φ and for values of $\Phi \rightarrow \pi/2$, the Faraday term vanishes. In this case the condition $M_F \gg |\Delta M|$ would not be valid anymore. Therefore, we focus

⁹It is import to stress that this approximation does not imply that there are no excitations in the background fields.

¹⁰The Stokes ‘vector’ defined here is not really a vector in the mathematical sense since its components do not transform as those of an usual vector under coordinate transformation. The letter S used from now on for the Stokes vector should not be confused with the letter used to denote the photon-pseudoscalar particle system S of the previous section.

on for the moment in the case when $\Phi \neq \pi/2$ in such way that condition $M_F \gg |\Delta M|$ holds and split the matrix $A(T)$ in the following way

$$A(T) = A_0(T) + \epsilon A_1(T) = \frac{1}{HT} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2M_F(T) & 0 \\ 0 & -2M_F(T) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tilde{G}(T) \\ 0 & 0 & \tilde{G}(T) & 0 \end{pmatrix}, \quad (4.3)$$

where we wrote $\Delta M(T)/(HT) = \epsilon \tilde{G}(T)$ and $\epsilon \ll 1$ is a parameter that depends on momentum k , magnetic field strength B_e and on the angle Φ . Here $\tilde{G}(T)$ is a function that depends only on the temperature T . The numerical factor in the product $\epsilon \tilde{G}(T)$ is included in the parameter ϵ . The expression of ϵ will be given in the next sections. The second term which appears in (4.2) corresponds to the Hubble friction and its contribution to $S(T)$ appears as a damping factor of the form $\exp[-3 \int_T (1/T') dT']$ and it is common to all components of the Stokes vector. The easiest way to see it, is by observing that matrix $A(T)$ commutes with $(3/T)\mathbf{I}$ for every T . For the moment we concentrate on the solution of Eq. (4.2) without the damping term and include it in the final result.

We look for solution of the Stokes vector up to first order in ϵ as follows

$$S(T) = S_0(T) + \epsilon S_1(T) + O(\epsilon^2) + \dots \quad (4.4)$$

Inserting expansion (4.4) into Eq. (4.2) and collecting the appropriate terms we get the following matrix equations

$$S'_0(T) = A_0(T)S_0(T), \quad (4.5)$$

$$S'_1(T) = A_0(T)S_1(T) + A_1(T)S_0(T). \quad (4.6)$$

We may observe that for different cosmological temperatures, the commutator of $[A_0(T_1), A_0(T_2)] = 0$ which allows us to find the following exact solution for $S_0(T)$

$$S_0(T) = \begin{pmatrix} \cos[F(T)] & -\sin[F(T)] & 0 \\ \sin[F(T)] & \cos[F(T)] & 0 \\ 0 & 0 & 1 \end{pmatrix} S_0(T_i),$$

where T_i is the initial temperature and

$$F(T) \equiv 2 \int_T^{T_i} \frac{M_F(T')}{H(T')T'} dT', \quad (4.7)$$

where $T \leq T_i$ is the CMB temperature after the decoupling time. We may observe that the homogeneous part of Eq. (4.6) has the same solution as Eq. (4.5) with the replacement $S_0(T) \rightarrow S_1(T)$. The non homogeneous part of Eq. (4.6) can be solved with the method of the variations of constants. Performing several algebraic operations that involve matrix exponentiations, collecting all the appropriate terms together and including the term corresponding to the Hubble friction we get the following solutions for

the components of the Stokes vector to the first order in ϵ :

$$(T_i/T)^3 I(T) = I_i, \quad (4.8)$$

$$(T_i/T)^3 Q(T) = \cos[F(T)]Q_i - \sin[F(T)]U_i + \left(\cos[F(T)] \int_T^{T_i} \epsilon \tilde{G}(T') \sin[F(T')] dT' \right. \\ \left. - \sin[F(T)] \int_T^{T_i} \epsilon \tilde{G}(T') \cos[F(T')] dT' \right) V_i, \quad (4.9)$$

$$(T_i/T)^3 U(T) = \sin[F(T)]Q_i + \cos[F(T)]U_i + \left(\sin[F(T)] \int_T^{T_i} \epsilon \tilde{G}(T') \sin[F(T')] dT' \right. \\ \left. + \cos[F(T)] \int_T^{T_i} \epsilon \tilde{G}(T') \cos[F(T')] dT' \right) V_i, \quad (4.10)$$

$$(T_i/T)^3 V(T) = - \left(\int_T^{T_i} \epsilon \tilde{G}(T') \sin[F(T')] dT' \right) Q_i - \left(\int_T^{T_i} \epsilon \tilde{G}(T') \cos[F(T')] dT' \right) U_i + V_i, \quad (4.11)$$

where I_i, Q_i, U_i, V_i are the values of the Stokes parameters at temperature $T = T_i$.

There are several interesting considerations that can be made about (4.9)-(4.11). In the first place we may notice that each solution is proportional to the initial values of the Stokes parameters Q_i, U_i and V_i , as one would expect from a first order system of linear differential equations. This implies that if the initial conditions are all zero, as for example in the case of unpolarized light, it would remain unpolarized during the universe expansion. If this is the case, the Faraday effect, the vacuum polarization and CM effect would not have any impact on the CMB polarization at all. The only way that these effects can have an impact on the CMB polarization, would be if the CMB is initially polarized. As already mentioned in the introduction section, Thomson scattering would generate CMB linear polarization only if there are anisotropies in the CMB temperature (or intensity). If the incident light is initially unpolarized and anisotropic, Thomson scattering generates outgoing polarized light with non zero Stokes parameters I and Q while $V = U = 0$. This is a general property of Thomson scattering for anisotropic incident light. Since the parameters Q and U depends on the coordinate system, one can rotate the system to a common one, in such a way to have $U \neq 0$. It can be shown that in the rotated system, the temperature anisotropy of the CMB generates non zero initial Stokes parameters Q_i and U_i at the decoupling time [16], [32]

$$Q_i = \frac{3\sigma_T}{4\pi\sigma_B} \sqrt{\frac{2\pi}{15}} \text{Re } a_{22}, \quad U_i = -\frac{3\sigma_T}{4\pi\sigma_B} \sqrt{\frac{2\pi}{15}} \text{Im } a_{22}, \quad (4.12)$$

where σ_T is the Thomson scattering cross section, σ_B is the cross sectional area of the scattered light and a_{22} is the second multipole coefficient used in expanding the incident photon intensity in spherical harmonics Y_{lm} . We have intentionally labelled with i the values of Q and U at the decoupling time and use them as the initial conditions in (4.9)-(4.11). However, as it has been well studied in the literature, Thomson scattering does not generate circular polarization and in the case of CMB this fact is confirmed since equation which governs evolution of V parameter due to Thomson scattering has no source term [16]. Consequently in this section we assume that at decoupling $V_i = 0$.

In the second place we may note from (4.9)-(4.10) that due to magneto-optic effects, light during its propagation contemporary has its polarization plane rotated and there is generation of phase shift between photon states A_+ and A_- . This behaviour is well know in optics and the medium which induces such effects is usually referred as rotated retarder, namely it rotates the polarization plane and generates ellipticity at the same time. We may note that for $V_i = 0$ the expressions for Q and U are the same as in the case of solely Faraday effect taken into account. Moreover, as it has been discussed in Sec. 2, in case when light is linearly polarized, $\Delta M(T) = 0$ the Faraday effect alone does not generate circular polarization. Also we may note the contribution of the Hubble friction term to the Stokes parameters, namely $(T/T_i)^3$ that for convenience reasons we putted it on the r. h. s. of (4.8)-(4.11). The *effective* scaling of the Stokes parameters due to universe expansion is not $(T/T_i)^3$ but $(T/T_i)^2$ since the Stokes

parameters contain an intrinsic scaling of (T/T_i) . This is due to the fact that in the WKB approximation the fields have an intrinsic normalization of $1/\sqrt{\omega(t)}$ with $\omega(t)$ being photon/pseudoscalar energy. We did not show explicitly this factor for simplicity which gives a multiplicative factor to fields proportional to $T^{1/2}$ and to the Stokes parameters proportional to T . Since the scaling factor due to universe expansion is common to all Stokes parameters and because we are mostly interested in their ratio or expressions that contain their ratio, such as polarization degrees $P_{L,C}$, this term eventually cancels out. For example, the degree of linear polarization of the CMB remains constant during universe expansion to first order in ϵ

$$P_L(T) = \frac{\sqrt{Q^2(T) + U^2(T)}}{I(T)} = \sqrt{Q_i^2 + U_i^2} = P_L(T_i),$$

where we took for simplicity $I_i = 1$ and $V_i = 0$ in (4.9)-(4.10). The total rotation angle of linear polarization of the CMB due to the Faraday effect is given by $\psi_F(T) = F(T)/2$. The contribution of CM and vacuum polarization effects to linear polarization does not appear to the first order in ϵ . Their contribution appears only to second order in ϵ but for our purposes only expansion to first order is important in this section.

Apart the fact that the polarization plane of the CMB is rotated due to the Faraday effect, another interesting effect is the generation of circular polarization with non zero Stokes parameter $V(T)$. Even in the case when there is not circular polarization at the decoupling time, it is generated afterwards due to vacuum polarization and CM effects. In the case of vanishing V_i we have

$$(T_i/T)^3 V(T) = -Q_i \int_T^{T_i} \epsilon \tilde{G}(T') \sin[F(T')] dT' - U_i \int_T^{T_i} \epsilon \tilde{G}(T') \cos[F(T')] dT'. \quad (4.13)$$

Based on (4.13), in this section we concentrate mostly in calculation of degree of circular polarization of the CMB in cases of vacuum polarization and CM effects. Their contribution is included in the term $\Delta M(T)$ where $\Delta M(T) = \Delta M_{\text{CM}}(T) + \Delta M_{\text{QED}}(T)$.

In both terms on the r. h. s. of (4.13) enters the function $F(T)$ which represents the effect of the Faraday effect. To have an analytic expression for $F(T)$ we need first the expression for $M_F(T)$ which is given by one of the off-diagonal terms of Π^{ij} . The Faraday effect is induced by the longitudinal component of the magnetic field with respect to \mathbf{k} , namely by $\mathbf{B}_L = \mathbf{B}_e \cos(\Phi)$. Consequently, linearly polarized electromagnetic wave propagating along the direction of the external magnetic field, has its polarization plane rotated with an angle proportional to B_L . This occurs because the right and left handed indexes of refraction n_R and n_L are different from each other, which make possible mixing between linearly polarized states A_+ and A_\times . The expression for the Faraday term is given by

$$M_F = |\Pi^{12}|/2\omega = \frac{\omega_{\text{pl}}^2 \omega_c \cos(\Phi)}{2(\omega^2 - \omega_c^2)},$$

where $\omega_{\text{pl}}^2 = 4\pi\alpha n_e/m_e$ is the plasma frequency, n_e is the free electron number density, m_e is the electron mass and we used $k \simeq \omega$ for photons. Here $\omega_c = eB_e/m_e$ is the cyclotron frequency with e being the electron charge. During propagation of the electromagnetic wave in a magnetized medium, the wave polarization remains unchanged for initial linearly polarized wave, but the linear polarized states A_+ and A_\times propagate with a new index of refraction Δn_F in the medium which is given by $\Delta n_F = n_R - n_L$.

The last thing that remains to calculate is the expression for the Hubble parameter which enters $F(T)$ in (4.7) and in $\epsilon\tilde{G}(T)$. In general, its expression in the case of zero spatial curvature ($\kappa = 0$) is given by

$$H(T) = H_0 \left(\Omega_\Lambda + \Omega_M (T/T_0)^3 + \Omega_R (T/T_0)^4 \right)^{1/2},$$

where H_0 is the Hubble parameter at the present epoch, $H_0 = H(T_0)$, Ω_Λ is the density parameter of the vacuum energy, Ω_M is the matter density parameter and Ω_R is the density parameter of relativistic particles. According to the Planck collaboration [33], values of density parameters of nonrelativistic matter and vacuum energy are respectively $h_0^2 \Omega_M = 0.12$ and $\Omega_\Lambda = 0.68$ with $h_0 = 0.67$. The density

parameter of relativistic particles it is straightforward to calculate, $\Omega_R = 4.15 \times 10^{-5} h_0^{-2}$ which includes the contribution of photons and three neutrino species assumed to be nearly massless. The contribution of the external magnetic field to the energy density budget of relativistic fields can be safely neglected since its energy density is $\rho_B(T_0) \simeq 10^{-7} (B_0/\text{nG})^2 \rho_\gamma(T_0)$.

4.1 Vacuum polarization in external magnetic field

Having the expressions for the Faraday term $M_F(T)$ and $H(T)$, we have almost all necessary ingredients to calculate the degree of circular polarization¹¹ at present time, $V(T_0)$. Vacuum polarization and CM effects are responsible for generation of circular polarization. They are induced by the transverse component of the external magnetic field, B_T . In both effects the linear polarization indexes of refraction, n_+ and n_\times are different from each other. Contrary to the Faraday effect which has its index of refraction proportional to B_L , vacuum polarization and CM effects have their indexes of refraction proportional to B_T^2 .

In this section we consider the contribution of vacuum polarization¹² to $V(T)$ separately from the CM effect which will be considered in the next section. Vacuum polarization occurs (not only) in the presence of an external magnetic field due to creation of electron/positron pair from the vacuum, see Fig. 2. The expressions of elements of photon polarization tensor¹³ corresponding to the states A_+ and A_\times in case of vacuum polarization, for slowly varying external magnetic field in space and time over the Compton wavelength, are respectively given by [34]

$$\Pi_{\text{QED}}^{22} = -4\kappa \omega^2 \sin^2(\Phi), \quad \Pi_{\text{QED}}^{11} = -7\kappa \omega^2 \sin^2(\Phi), \quad (4.14)$$

where $\kappa = (\alpha/45\pi)(B_e/B_c)^2$ and $B_c = m_e^2/e$ is the critical magnetic field. Using (4.14) and definitions of M_+ and M_\times we get

$$\Delta M_{\text{QED}} = -\frac{3}{2}\kappa \omega \sin^2(\Phi). \quad (4.15)$$

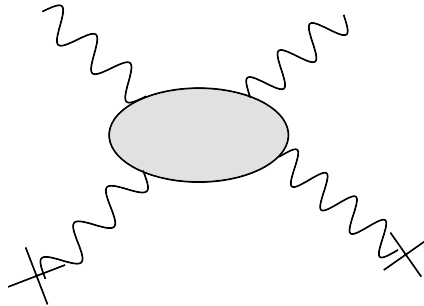


Figure 2: Vacuum polarization in an external magnetic field. The cross vertexes denote external magnetic field and the wavy lines denote photons. In calculating the polarization tensor for the vacuum polarization, only the contribution of electron/positron loop is included.

Now using the definition of κ and taking into account that the photon energy scales with the temperature as $\omega = \omega_0(T/T_0)$ and assuming magnetic field flux conservation in the cosmological plasma with

¹¹The degree of circular polarization as discussed in Sec. 2 is defined as $P_C = |V|/I$. Since we choose $I_i = 1$ and because the scaling term due to universe expansion cancels out, $P_C = |V|$. Consequently in this section we calculate only $V(T)$ and take its absolute value if it is a negative quantity.

¹²Vacuum polarization in external magnetic field is a non linear QED effect which Lagrangian density is given by the Euler-Heisenberg term $\mathcal{L}_{EH} = \frac{\alpha^4}{90m_e^4} \left[(F_{\mu\nu}F^{\mu\nu})^2 + \frac{7}{4}(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 \right]$.

¹³The diagonal terms of the polarization tensor include the contribution of plasma effects, vacuum polarization and CM effect. Since plasma effects are the same for A_+ and A_\times and because in this section and in the next we calculate, $\Delta M = M_+ - M_\times$, the plasma term cancels out.

$B_e(T) = B_e(T_0)(T/T_0)^2$ we get

$$\begin{aligned} G(T) &= \epsilon_{\text{QED}} \tilde{G}(T) = -8.12 \times 10^{-14} \left(\frac{\nu_0}{\text{Hz}} \right) \left(\frac{B_{e0}}{G} \right)^2 \left(\frac{T}{T_0} \right)^{5/2} \sin^2(\Phi) \quad (\text{K}^{-1}), \\ F(T) &= 8.71 \times 10^{25} \cos(\Phi) \left(\frac{\text{Hz}}{\nu_0} \right)^2 \left(\frac{B_{e0}}{G} \right) \int_T^{T_i} X_e(T') \left(\frac{T'}{T_0} \right)^{1/2} dT' \quad (\text{K}^{-1}), \end{aligned} \quad (4.16)$$

where T_0 is the CMB temperature today and $\omega_0 = 2\pi\nu_0$ with ν_0 being the CMB frequency at present. Deriving (4.16) we used the fact that $\omega_c \ll \omega$ in the Faraday term, expressed the free electron number density $n_e(T)$ which enters the plasma frequency as $n_e(T) = 0.76 n_B(T_0) X_e(T) (T/T_0)^3$ with $X_e(T)$ being the ionization fraction of free electrons and $n_B(T_0)$ is the baryon number density at present. Moreover, we assumed for simplicity that only non relativistic matter contributes to the Hubble parameter, namely $H(T) \simeq H_0 \sqrt{\Omega_M} (T/T_0)^{3/2}$. Until now we did not give any explicit expression for the parameter ϵ_{QED} . This parameter can be extracted immediately from $G(T)$ in (4.16) and it is given by

$$\epsilon_{\text{QED}} \equiv -8.12 \times 10^{-14} \left(\frac{\nu_0}{\text{Hz}} \right) \left(\frac{B_{e0}}{G} \right)^2 T_0^{-5/2} \sin^2(\Phi) \quad (\text{K}^{-1}).$$

With expressions of $F(T)$, $G(T)$ and ϵ_{QED} we have all necessary quantities to calculate $V(T)$ in (4.13). Let us start with the first term on the r. h. s. of (4.13) which has the following dependence on the temperature

$$\int_T^{T_i} \epsilon_{\text{QED}} \tilde{G}(T') \sin[F(T')] dT' = \epsilon_{\text{QED}} \int_T^{T_i} T'^{5/2} \sin \left[\rho \int_{T'}^{T_i} X_e(T'') T''^{1/2} dT'' \right] dT', \quad (4.17)$$

where we have defined

$$\rho \equiv 8.71 \times 10^{25} \cos(\Phi) \left(\frac{\text{Hz}}{\nu_0} \right)^2 \left(\frac{B_{e0}}{G} \right) T_0^{-1/2} \quad \text{K}^{-1}.$$

As one would expect there is an integration in $X_e(T)$ on the r. h. s. of expression (4.17) that complicates the situation quite a lot. Indeed, there is no known analytic expression for $X_e(T)$ which in general satisfies a complicated differential equation, see Ref. [35] for details. At the temperature $T \simeq 3000$ K numerical solution of the equation satisfied by $X_e(T)$, shows that ionization fraction is $X_e \simeq 0.13$ and drops down to $X_e \simeq 2 \times 10^{-2}$ at the temperature $T = 2000$ K. When the temperature is about 200 K it drops down to $X_e \simeq 2.7 \times 10^{-4}$ and remains almost constant afterwards if no reionization epoch is assumed. In this work we use the solution for $X_e(T)$ given in Ref. [35] and interpolate it with $X_e \simeq 1$ for $T \lesssim 21.8$ K which corresponds to the period of end of the reionization epoch.

The integral in (4.17) has analytic solution in terms of the incomplete Euler gamma functions, if the expression for $X_e(T)$ is constant¹⁴. However, one may observe that for $T \leq 2970$ K, for magnetic field strength $B_{e0} \leq 1$ nG and frequencies $\nu_0 \geq 10^{10}$ Hz, the expression inside sine function is less than unity, namely $F(T) < 1$. In this case one can use series expansion and consider only the first term. Consequently we obtain

$$\int_T^{T_i} \epsilon_{\text{QED}} \tilde{G}(T') \sin[F(T')] dT' \simeq \epsilon_{\text{QED}} \rho \int_T^{T_i} T'^{5/2} \int_{T'}^{T_i} X_e(T'') T''^{1/2} dT'' dT'. \quad (4.18)$$

We are interested in calculating the integral in (4.18) at $T = T_0$ and numerical calculation gives

$$\int_{T_0}^{T_i} T'^{5/2} \int_{T'}^{T_i} X_e(T'') T''^{1/2} dT'' dT' \simeq 5.17 \times 10^{14} \quad (\text{K}^5),$$

¹⁴In principle one can obtain analytic solution for the integral (4.17) by considering the average value of $X_e(T)$ at the post decoupling epoch as we shown for some cases in Sec. 6. However, we don't need to do it here because the vacuum polarization dominates in general the CM effect for $\nu_0 > 10^{10}$ Hz. Consequently, we can get more accurate result by considering the numerical solution for X_e , expand the argument of sine function and integrate it numerically.

and expression (4.18) becomes

$$\int_{T_0}^{T_i} \epsilon_{\text{QED}} \tilde{G}(T') \sin[F(T')] dT' = 5.17 \times 10^{14} \epsilon_{\text{QED}} \rho \quad (\text{K}^5). \quad (4.19)$$

Now it remains to calculate the second term on the r. h. s. of (4.13). Based on the same arguments as we did above for the first term, the argument of cosine function is smaller than unity and we can write

$$\int_{T_0}^{T_i} \epsilon_{\text{QED}} \tilde{G}(T') \cos[F(T')] dT' \simeq \frac{2}{7} \epsilon_{\text{QED}} \left(T_i^{7/2} - T_0^{7/2} \right).$$

The value of V at present time would be for $T_i = 2970$ K (the CMB temperature at the redshift $1+z=1090$ corresponding to the decoupling time) and $T_0 = 2.725$ K

$$V_0(\nu_0, B_0, \Phi) \simeq 1.8 \times 10^{26} \sin^2(\Phi) \cos(\Phi) \left(\frac{\text{Hz}}{\nu_0} \right) \left(\frac{B_{e0}}{G} \right)^3 Q_i + 2.7 \times 10^{-3} \sin^2(\Phi) \left(\frac{\nu_0}{\text{Hz}} \right) \left(\frac{B_{e0}}{G} \right)^2 U_i. \quad (4.20)$$

If for example we take $\nu_0 = 30$ GHz and $B_{e0} = 1$ nG, we get

$$V_0(\Phi) \simeq 6 \times 10^{-12} \sin^2(\Phi) \cos(\Phi) Q_i + 8.1 \times 10^{-11} \sin^2(\Phi) U_i. \quad (4.21)$$

We have checked that numerical values derived from expression (4.20), perfectly agree with numerical solutions in the case when one assumes $H \simeq H_0 \sqrt{\Omega_M} (T/T_0)^{3/2}$. The difference between numerical solutions with total H and semi-analytic solutions with only matter contribution to H , is that in the former case numerical solutions are in general smaller by a factor less than $\sqrt{2}$ with respect to the latter case. We may note that the second term in (4.21) is proportional to the frequency and for higher values of ν_0 , V_0 increases linearly with ν_0 .

So far, we have derived our results in the case when $\Phi \neq \pi/2$, which allowed us to find perturbative solution for the Stokes vector $S(T)$. In the case when the magnetic field is transverse, this approximation is not valid anymore since $M_F(T) \rightarrow 0$ for $\Phi \rightarrow \pi/2$. However, if $\Phi = \pi/2$ it is not necessary to work with the perturbative approach since the equation for the Stokes vector simplifies significantly. Indeed, for $\Phi = \pi/2$ there is only mixing between the Stokes parameters U and V . The solution of equations of motion for the Stokes parameters U and V , in transverse magnetic field are immediate and read

$$\begin{aligned} Q(T) &= Q_i, & U(T) &= \cos[\mathcal{G}(T)] U_i + \sin[\mathcal{G}(T)] V_i, \\ V(T) &= -\sin[\mathcal{G}(T)] U_i + \cos[\mathcal{G}(T)] V_i, \end{aligned} \quad (4.22)$$

where we have defined $\mathcal{G}(T) = \int_T^{T_i} G(T') dT'$.

In case of $V_i = 0$, one would get for V

$$V(T) = -\sin[\mathcal{G}(T)] U_i.$$

In order to estimate $V(T)$ at present time for $\Phi = \pi/2$, we first must calculate $\mathcal{G}(T)$ in the argument of sine function. Consequently, we get

$$\mathcal{G}(T) = -2.32 \times 10^{-14} T_0^{-5/2} \left(\frac{\nu_0}{\text{Hz}} \right) \left(\frac{B_{e0}}{G} \right)^2 \left(T_i^{7/2} - T^{7/2} \right) \quad (\text{K}^{-1}),$$

and the value of V at $T = T_0$ is

$$V_0(\nu_0, B_{e0}) \simeq \sin \left[2.7 \times 10^{-3} \left(\frac{\nu_0}{\text{Hz}} \right) \left(\frac{B_{e0}}{G} \right)^2 \right] U_i. \quad (4.23)$$

In general for a wide range of the parameters ν_0 and B_{e0} in (4.23), the argument of sine function is much less than unity and one can replace to first order the sine with its argument. If we take for example the values $\nu_0 = 100$ GHz and $B_{e0} = 1$ nG we get

$$V_0 \simeq 2.7 \times 10^{-10} U_i.$$

If the magnetic field is 10 nG we would get $V_0 \simeq 2.7 \times 10^{-8} U_i$. It is worth to note that in case of transverse magnetic field, the vacuum polarization induces also a rotation of the polarization plane. Indeed, as can be inferred from (4.22), the rotation angle of the polarization plane is given by

$$\tan[2\psi(T)] = \tan[2\psi(T_i)] \cos[\mathcal{G}(T)], \quad (4.24)$$

which in general is a very small quantity for vacuum polarization.

Until now we kept the dependence on Φ explicit in (4.20) but it is more convenient to average over all possible orientations of \mathbf{B}_e relative to \mathbf{k} . We must note that (4.20) has been derived by assuming $\Phi \neq \pi/2$ which allowed us to find perturbative solution for the Stokes vector $S(T)$. However, we may note that in the limit $\Phi \rightarrow \pi/2$, the first term in (4.20) goes to zero while the second term coincides with the argument of sine function in (4.23) which has been found exactly. This fact allows us, to find the following expression for the rms of V_0 in (4.20), which for $F(T) < 1$ is given by

$$\langle V_0^2(\nu_0, B_{e0}) \rangle^{1/2} \simeq \left[2 \times 10^{51} \left(\frac{\text{Hz}}{\nu_0} \right)^2 \left(\frac{B_{e0}}{\text{G}} \right)^6 Q_i^2 + 2.73 \times 10^{-6} \left(\frac{\nu_0}{\text{Hz}} \right)^2 \left(\frac{B_{e0}}{\text{G}} \right)^4 U_i^2 \right]^{1/2}. \quad (4.25)$$

Assuming for example, $B_{e0} = 1$ nG, $Q_i \simeq U_i$ we get for $\nu_0 = 30$ GHz and $\nu_0 = 700$ GHz respectively $\langle V_0^2 \rangle^{1/2} \simeq 4.96 \times 10^{-11} Q_i$ and $\langle V_0^2 \rangle^{1/2} \simeq 1.15 \times 10^{-9} Q_i$. We may note from (4.25), that biggest contribution comes for values of $\Phi \rightarrow \pi/2$ or transverse fields and for higher values of B_{e0} , the rms of V_0 is bigger.

Another interesting case is when the arguments of sine and cosine functions in (4.22) are equal to $\pi/2$, namely $\mathcal{G}(T) = \pi/2$. This condition is fulfilled when

$$\left(\frac{\nu_0}{\text{Hz}} \right) = 581.3 \left(\frac{\text{G}}{B_{e0}} \right)^2. \quad (4.26)$$

When this condition is met, we would have $V(T) = U_i$. However, in order for condition (4.26) to be fulfilled, the value of ν_0 must be much far beyond the present observed CMB spectrum for reasonable values of B_{e0} .

4.2 The Cotton-Mouton effect

As briefly mentioned in the previous sections, the CM effect is a birefringence effect which is induced in a medium in presence of transverse external magnetic field and which generates elliptic polarization. This effect has been studied and experimented in gases and liquids where limits on the CM constant C_{CM} are set or its value is established. However, the CM effect does not only manifest in gases and liquids but also in plasma¹⁵. The theory of this phenomena is studied to some extent classically and also quantum mechanically and for a discussion on this mechanism see Ref. [1].

After the decoupling epoch, the ionization fraction of free electrons rapidly dropped down to an almost constant value of $X_e \simeq 2.7 \times 10^{-4}$ and later again it increased to $X_e \simeq 1$ at the reionization epoch. On the other hand almost all baryons would bind together to form the light elements such as atomic hydrogen and helium etc. This state of mixed hydrogen and helium gas (plus a small fraction of other light elements) with the electron plasma continued coexisting till the start of the reionization

¹⁵I learned only recently about the CM effect in plasma.

epoch. In order to study the impact of CM effect in the generation of CMB polarization, we need the value of ΔM_{CM} for hydrogen and helium gas and for the electron plasma.

Theory of CM effect in gases has been extensively studied in the literature and for a review on the subject see Ref. [36] and references there. In case of gases, theoretical calculations give the following expression for the difference in index of refraction $\Delta n_{\text{CM}}^{\text{gas}}$ [36]

$$\Delta n_{\text{CM}}^{\text{gas}} = \pi B_e^2 \sin^2(\Phi) n^{\text{gas}} \Delta\eta / (4\pi\epsilon_0), \quad (4.27)$$

where n^{gas} is the gas number density and $\Delta\eta$ is called the hypermagnetizability anisotropy. Here it is assumed that n^{gas} obeys the perfect gas law or its closely related ideal gas law. In general $\Delta\eta$ will depend on the type of gas and on the incident energy of the electromagnetic radiation. Quite often the CM constant is also defined through the relation

$$\Delta n_{\text{CM}}^{\text{gas}} = C_{\text{CM}} \lambda B_e^2 \sin^2(\Phi), \quad (4.28)$$

where λ is the wave length of the incident electromagnetic wave (not to be confused with photon helicity state). By comparing (4.28) with (4.27) we get the following expression for $C_{\text{CM}} = \pi \Delta\eta n^{\text{gas}} / \lambda$.

In our case, we are interested in only the magnetic hypermagnetizability of hydrogen and helium gases since these elements are the most abundant ones and neglect the contribution of the other light elements. Following Ref. [36], theoretical values of $\Delta\eta$ in the limit of zero incident photon momentum, gas temperature $T_{\text{gas}} = 273.15$ K and gas pressure $P_{\text{gas}} = 1$ atm are respectively given by $\Delta\eta_{\text{H}} = 13.33$ au and $\Delta\eta_{\text{He}} = 1.06$ au where 1 au of η is $\simeq 2.682 \times 10^{-44} (4\pi\epsilon_0) \text{ G}^{-2} \text{ cm}^{-3}$ [36] with $\epsilon_0 = 1$ in the rationalized Lorentz-Heaviside system. Consequently we get

$$\Delta M_{\text{CM}}^{\text{gas}} \simeq \pi \omega B_e^2 \sin^2(\Phi) (Y_{\text{H}} \Delta\eta_{\text{H}} + Y_{\text{He}} \Delta\eta_{\text{He}}) n_B / (4\pi\epsilon_0),$$

where Y_{H} , Y_{He} are respectively the primordial abundances of atomic hydrogen and helium. Assuming that $\Delta\eta$ does not change significantly in the frequency range corresponding to the CMB after the decoupling epoch, we get

$$\Delta M_{\text{CM}}^{\text{gas}}(T) = 1.2 \times 10^{-60} \left(\frac{\nu_0}{\text{Hz}} \right) \left(\frac{B_{e0}}{\text{G}} \right)^2 \sin^2(\Phi) \left(\frac{T}{T_0} \right)^8 \quad (\text{K}). \quad (4.29)$$

The contribution of the electron plasma to the CM effect enters the diagonal elements of the polarization tensor in magnetized plasma, Π^{11} and Π^{22} . The difference with respect to the Faraday effect is that CM effect is quadratic in the amplitude of transverse magnetic field and one would expect that for typical values of the cosmic magnetic field, its magnitude would be much weaker than the Faraday effect. In general, for a magnetized plasma the contribution of the CM effect to the photon polarization tensor is given by [27]

$$\Pi_{\text{CM}}^{11} = \frac{\omega^2 \omega_{\text{pl}}^2}{\omega^2 - \omega_c^2} - \frac{\omega_{\text{pl}}^2 \omega_c^2}{\omega^2 - \omega_c^2} \sin^2(\Phi), \quad \Pi_{\text{CM}}^{22} = \frac{\omega^2 \omega_{\text{pl}}^2}{\omega^2 - \omega_c^2}. \quad (4.30)$$

We may note that in (4.30), the CM term appears only in Π^{11} (the second term) while it does not appear in Π^{22} . This is due to the fact that we have choosen since the beginning the transverse part of B_e along the x axis with no y component. Using the definition of ΔM , for the CM effect in magnetized plasma, we get

$$\Delta M_{\text{CM}}^{\text{pl}} = - \frac{\omega_{\text{pl}}^2 \omega_c^2}{2\omega(\omega^2 - \omega_c^2)} \sin^2 \Phi.$$

In case of CMB, we have that photon frequency is much bigger than cyclotron frequency and we can approximate $\omega^2 - \omega_c^2 \simeq \omega^2$. Consequently, we get

$$\Delta M_{\text{CM}}^{\text{pl}} = -2.82 \times 10^3 \left(\frac{\text{Hz}}{\nu_0} \right)^3 \left(\frac{B_{e0}}{\text{G}} \right)^2 X_e(T) \left(\frac{T}{T_0} \right)^4 \sin^2(\Phi) \quad (\text{K}). \quad (4.31)$$

If we compare (4.31) with (4.29), we may observe that for the parameter space of magnetic field amplitude B_{e0} and photon frequency ν_0 of interest, the contribution of hydrogen and helium gases to the CM effect is much smaller than the contribution of electron plasma. Therefore from now we will neglect the gas contribution to the CM effect.

Now we can calculate the contribution of CM effect to $V(T)$ in the same way as we did for the vacuum polarization. In case when $\Phi \neq \pi/2$ and $F(T) < 1$ at post decoupling epoch, we have

$$\begin{aligned} \int_{T_0}^{T_i} G(T') \sin[F(T')] dT' &\simeq \epsilon_{\text{CM}} \rho \int_{T_0}^{T_i} T'^{3/2} X_e(T') \int_{T'}^{T_i} X_e(T'') T''^{1/2} dT'' dT', \\ &= 3.46 \times 10^9 \epsilon_{\text{CM}} \rho \quad (\text{K}^4), \end{aligned} \quad (4.32)$$

where in the second term in (4.32) numerical integration has been used and defined $G(T) = \epsilon_{\text{CM}} \tilde{G}(T)$ with ϵ_{CM}

$$\epsilon_{\text{CM}} = -1.21 \times 10^{32} \left(\frac{\text{Hz}}{\nu_0} \right)^3 \left(\frac{B_{e0}}{\text{G}} \right)^2 T_0^{-3/2} \sin^2(\Phi) \quad (\text{K}^{-1}).$$

The second term that enters the r. h. s. of (4.13) is given by

$$\int_{T_0}^{T_i} G(T') \cos[F(T')] dT' \simeq \epsilon_{\text{CM}} \int_{T_0}^{T_i} T'^{3/2} X_e(T') dT' = 4.45 \times 10^6 \epsilon_{\text{CM}} \quad (\text{K}^{5/2}).$$

The total expression for the degree of circular polarization $V(T)$ at present time is given by

$$V_0(\omega_0, B_{e0}, \Phi) \simeq -3.46 \times 10^9 \epsilon_{\text{CM}} \rho Q_i - 4.45 \times 10^6 \epsilon_{\text{CM}} U_i. \quad (4.33)$$

Now we can put some numbers in (4.33) in order to estimate V_0 . For example in the case when $\nu_0 = 30$ GHz and $B_{e0} = 1$ nG we get

$$V_0 = 2 \times 10^{-13} \sin^2(\Phi) \cos(\Phi) Q_i + 4.43 \times 10^{-12} \sin^2(\Phi) U_i. \quad (4.34)$$

The case when the magnetic field is completely transverse, $\Phi = \pi/2$ is treated in the same way as in the case of vacuum polarization. What we need is to calculate $\mathcal{G}(T)$ in the case of CM effect, which in most cases is $\ll 1$. We remind that expressions for the Stokes parameters in case of transverse magnetic field are found exactly without using perturbation theory and are given in (4.22). In case when $V_i = 0$, from (4.22) we get

$$Q_0 = Q_i, \quad U_0 \simeq U_i, \quad V_0 \simeq 1.2 \times 10^{38} \left(\frac{\text{Hz}}{\nu_0} \right)^3 \left(\frac{B_{e0}}{\text{G}} \right)^2 U_i, \quad (4.35)$$

where we kept only the first order term in $\mathcal{G}(T) \ll 1$. The most interesting fact, is the relation $V_0 \propto \nu_0^{-3}$ in (4.35). If we consider for example $B_{e0} = 1$ nG and $\nu_0 = 10^8$ Hz we get $V_0 \simeq 1.2 \times 10^{-4} U_i$ while for $\nu_0 = 10^9$ Hz we get $V_0 \simeq 1.2 \times 10^{-7} U_i$. In principle one can also calculate the rms of V_0 for the CM effect, as we did in case of vacuum polarization, but it is not that easy. Indeed, expression (4.33) has been derived in the approximation when $F(T) < 1$, which assuming that $B_{e0} \leq 1$ nG, it is satisfied for $\nu_0 > 10^{10}$ Hz. However, the biggest contribution in V_0 comes from the low frequency part as we saw for the case when $\Phi = \pi/2$. Instead of looking for analytic solution even when $F(T) < 1$ is not satisfied and after estimate the rms of V_0 , one possible way to circumvent this situation, is to note that rms of V_0 is bigger for values of $\Phi \rightarrow \pi/2$. Indeed, we have checked the numerical solution and found that value of V_0 for $\Phi \neq \pi/2$, fixed B_{e0} and ν_0 is much smaller than that of $\Phi = \pi/2$. Consequently, one can approximate to very good accuracy the rms of V_0 with its value at $\Phi = \pi/2$.

The low frequency part of the CMB, $\nu_0 \sim 10^8$ Hz for $\Phi = \pi/2$ is very interesting since significant rotation of the polarization plane occurs. Indeed, if one keeps $\mathcal{G}(T) \ll 1$ up to second order in $U(T)$ in

expression (4.22) and assuming that at decoupling time $Q_i \simeq -U_i$, one gets for the rotation angle the following expression

$$\delta\psi(T_0) \simeq 1.8 \times 10^{75} \left(\frac{\text{Hz}}{\nu_0} \right)^6 \left(\frac{B_{e0}}{\text{G}} \right)^4, \quad (4.36)$$

where we wrote $\psi(T) \simeq \psi(T_i) + \delta\psi(T)$ with $|\delta\psi| \ll 1$ and used (4.24). If we consider $B_{e0} = 1 \text{ nG}$, $\nu_0 \simeq 10^8 \text{ Hz}$ we get $\delta\psi(T_0) \simeq 1.8 \times 10^{-9} \text{ rad}$, for $B \simeq 10^{-8} \text{ G}$ and same frequency we would get $\delta\psi(T_0) \simeq 1.8 \times 10^{-5} \text{ rad}$ and for $B = 10^{-7} \text{ G}$ we would get $\delta\psi(T_0) = 0.18 \text{ rad}$.

5 Three state density matrix and generalized Stokes parameters

In Sec. 4 we derived temperature (or time) evolution of the Stokes parameters in case when only Faraday effect, vacuum polarization and CM effects were included in the equations of motion of the density matrix. However, there still remain one effect left which is the photon-pseudoscalar mixing in magnetic field. Including this new effect, the equations of motion of the density matrix become more involved and instead of four equations which had in Sec. 4, now we have nine of them. This can be verified by inserting the mixing matrix M and the damping matrix D in Eq. (3.5). After we get the following system of differential equations

$$\begin{aligned} \dot{\rho}_{11} &= -M_F(\rho_{12} + \rho_{21}) - 3H\rho_{11}, \\ \dot{\rho}_{12} &= -M_F(\rho_{22} - \rho_{11}) + i(M_+ - M_\times)\rho_{12} - M_{\phi\gamma}\rho_{13} - 3H\rho_{12}, \\ \dot{\rho}_{13} &= -M_F\rho_{23} - i(M_\phi - M_+)\rho_{13} + M_{\phi\gamma}\rho_{12} - 3H\rho_{13}, \\ \dot{\rho}_{21} &= \rho_{12}^*, \\ \dot{\rho}_{22} &= M_F(\rho_{12} + \rho_{21}) - M_{\phi\gamma}(\rho_{32} + \rho_{23}) - 3H\rho_{22}, \\ \dot{\rho}_{23} &= M_F\rho_{13} + i(M_\times - M_\phi)\rho_{23} + M_{\phi\gamma}(\rho_{22} - \rho_{33}) - 3H\rho_{23}, \\ \dot{\rho}_{31} &= \rho_{13}^*, \\ \dot{\rho}_{32} &= \rho_{23}^*, \\ \dot{\rho}_{33} &= M_{\phi\gamma}(\rho_{23} + \rho_{32}) - 3H\rho_{33}, \end{aligned} \quad (5.1)$$

where the sign (*) means complex conjugate of a C-number and each element of the mixing matrix M depends on the cosmological time t , B_e and ν . We may observe that total intensity is diluted due to universe expansion only

$$\dot{\rho}_{11} + \dot{\rho}_{22} + \dot{\rho}_{33} = -3H(\rho_{11} + \rho_{22} + \rho_{33}),$$

which means that trace of the density matrix is not constant in time.

The system of Eq. (5.1) still is not in the desired form since the photon intensity, that in this section we denote with I_γ , is not a conserved quantity. As already discussed in Sec. 4, this is due to the fact that we are dealing with an open system interacting with the background. Even in the case of other magneto-optic effects which we treated in Sec. 4 there is interaction with the background, but with the difference that these effects conserve the photon number with momentum \mathbf{k} . Since I_γ is not conserved, it would be convenient to express the equations of elements of the density matrix in terms of generalized Stokes parameters¹⁶, that is extending the usual two state Stokes parameters to the case of three states.

The derivation of generalized Stokes parameters can be done in analogous way as one does with usual Stokes parameters. In Sec. 2, we expressed elements of two dimensional density matrix in terms of the Stokes parameters and one can check from direct calculation that expression (2.19) can be written in terms of the Pauli matrices σ_i as follows

$$\rho = \frac{1}{2} \left(S_0 \mathbf{I}_{2 \times 2} + \sum_{i=1}^3 S_i \sigma_i \right),$$

¹⁶Here ‘generalized Stokes parameters’ does not mean a generalization to the case of $n \in \mathbf{N}$ states but simply means going from the description of two state parameters to the three state parameters.

where we recall that $\langle S_0 \rangle = I_\gamma$, $\langle S_1 \rangle = U$, $\langle S_2 \rangle = V$, $\langle S_3 \rangle = Q$ and $\mathbf{I}_{2 \times 2}$ is the two dimensional identity matrix. The generalization of the usual two state Stokes parameters to the three state case can be done as follows

$$\langle \hat{S}_k \rangle := \text{Tr}(\rho \lambda_k), \quad (5.2)$$

where ρ is the 3×3 density matrix, \hat{S}_k (for $k \geq 1$) are the generators of SU(3) group and λ_k (for $k \geq 1$), ($k = 0, 1, \dots, 8$), are the so called Gell-Mann matrices

$$\begin{aligned} \lambda_0 &= \mathbf{I}_{3 \times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -i & 0 \\ i & 0 & 0 \end{pmatrix}, \\ \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \end{aligned}$$

Inserting λ_k into expression (5.2) we can get the explicit expressions for the generalized Stokes parameters. The first set of four Stokes parameters is given by (2.18) while the remaining set of parameters is given by

$$\begin{aligned} S_4 &= \rho_{13} + \rho_{31}, \quad S_5 = i(\rho_{13} - \rho_{31}), \quad S_6 = \rho_{23} + \rho_{32}, \\ S_7 &= i(\rho_{23} - \rho_{32}), \quad S_8 = \frac{1}{\sqrt{3}}(\rho_{11} + \rho_{22} - 2\rho_{33}). \end{aligned} \quad (5.3)$$

The corresponding Stokes operators to the set (5.3) in the basis $|A_+\rangle, |A_\times\rangle, |\phi\rangle$ are given by

$$\begin{aligned} \hat{S}_4 &= |A_+\rangle\langle\phi| + |\phi\rangle\langle A_+|, \quad \hat{S}_5 = i(|A_+\rangle\langle\phi| - |\phi\rangle\langle A_+|), \quad \hat{S}_6 = |A_\times\rangle\langle\phi| + |A_\times\rangle\langle\phi|, \\ \hat{S}_7 &= i(|A_\times\rangle\langle\phi| - |\phi\rangle\langle A_\times|), \quad \hat{S}_8 = \frac{1}{\sqrt{3}}(|A_+\rangle\langle A_+| + |A_\times\rangle\langle A_\times| - 2|\phi\rangle\langle\phi|). \end{aligned}$$

Having defined the generalized Stokes parameters, now we are at the position to parametrize three state density matrix in terms of them as follows

$$\rho = \frac{1}{2} \begin{pmatrix} \frac{2}{3}I + Q + \frac{1}{\sqrt{3}}S_8 & U - iV & S_4 - iS_5 \\ U + iV & \frac{2}{3}I - Q + \frac{1}{\sqrt{3}}S_8 & S_6 - iS_7 \\ S_4 + iS_5 & S_6 + iS_7 & \frac{2}{3}I - \frac{2}{\sqrt{3}}S_8 \end{pmatrix}, \quad (5.4)$$

where I is the total intensity which is given by $I = I_\gamma + I_\phi$. Using (5.4) we can write the system of Eqs. (5.1) as follows

$$\begin{aligned} \dot{I}_\gamma &= -M_{\phi\gamma}S_6 - 3HI_\gamma, \\ \dot{Q} &= -2M_FU + M_{\phi\gamma}S_6 - 3HQ, \\ \dot{U} &= 2M_FQ + (M_+ - M_\times)V - M_{\phi\gamma}S_4 - 3HU, \\ \dot{V} &= -(M_+ - M_\times)U - M_{\phi\gamma}S_5 - 3HV, \\ \dot{S}_4 &= -M_FS_6 + (M_+ - M_\phi)S_5 + M_{\phi\gamma}U - 3HS_4, \\ \dot{S}_5 &= -M_FS_7 - (M_+ - M_\phi)S_4 + M_{\phi\gamma}V - 3HS_5, \\ \dot{S}_6 &= M_FS_4 + (M_\times - M_\phi)S_7 + M_{\phi\gamma}(\sqrt{3}S_8 - Q) - 3HS_6, \\ \dot{S}_7 &= M_FS_5 - (M_\times - M_\phi)S_6 - 3HS_7, \\ \dot{S}_8 &= -\sqrt{3}M_{\phi\gamma}S_6 - 3HS_8. \end{aligned} \quad (5.5)$$

5.1 Equations of motion in absence of the Faraday effect

The system of Eqs. (5.5) is in the final form and we can immediately see from the equations of motion governing usual Stokes parameters, the contribution of the pseudoscalar field to the linear and circular polarization. Let us stress since now that an exact closed analytic solution for (5.5) is not possible. However, here we consider some particular cases, by using some reasonable approximations, which allow us to find semi-analytic solutions for Eqs. (5.5). Indeed, the system (5.5) can be simplified by considering the case of transverse external magnetic field, namely $\Phi = \pi/2$. This can be achieved by observing the CMB in the direction perpendicular to the external magnetic field and for this particular configuration, the Faraday effect would be completely absent.

In the case when the Faraday effect is absent, we get the following systems of decoupled differential equations in the variable T

$$\tilde{S}'_1(T) = B(T) \cdot \tilde{S}_1(T) + (3/T) \mathbf{I}_{3 \times 3} \tilde{S}_1(T), \quad \tilde{S}'_2(T) = C(T) \cdot \tilde{S}_2(T) + (3/T) \mathbf{I}_{3 \times 3} \tilde{S}_2(T), \quad (5.6)$$

where $\tilde{S}_1 = (U, V, S_4, S_5)^T$ and $\tilde{S}_2 = (I_\gamma, Q, S_6, S_7, S_8)^T$ are respectively two reduced (generalized) Stokes vectors¹⁷. The matrices B and C which enter Eqs. (5.6) are respectively given by

$$B(T) = \frac{1}{HT} \begin{pmatrix} 0 & -\Delta M & M_{\phi\gamma} & 0 \\ \Delta M & 0 & 0 & M_{\phi\gamma} \\ -M_{\phi\gamma} & 0 & 0 & -\Delta M_1 \\ 0 & -M_{\phi\gamma} & \Delta M_1 & 0 \end{pmatrix}, \quad C(T) = \frac{1}{HT} \begin{pmatrix} 0 & 0 & M_{\phi\gamma} & 0 & 0 \\ 0 & 0 & -M_{\phi\gamma} & 0 & 0 \\ 0 & M_{\phi\gamma} & 0 & -\Delta M_2 & -\sqrt{3}M_{\phi\gamma} \\ 0 & 0 & \Delta M_2 & 0 & 0 \\ 0 & 0 & \sqrt{3}M_{\phi\gamma} & 0 & 0 \end{pmatrix}.$$

5.2 Solution of first reduced Stokes vector \tilde{S}_1

Let us focus first on the solution of first reduced Stokes vector $\tilde{S}_1(T)$. We may note that an exact solution is not possible unless one uses some approximations that allow to find the solution by using perturbation theory, in a similar way as shown in Sec. 4. Therefore, we split the matrix $B(T)$ in the following order, $B(T) = B_1(T) + B_2(T)$

$$B_1 + B_2 = \frac{1}{HT} \begin{pmatrix} 0 & 0 & M_{\phi\gamma} & 0 \\ 0 & 0 & 0 & M_{\phi\gamma} \\ -M_{\phi\gamma} & 0 & 0 & 0 \\ 0 & -M_{\phi\gamma} & 0 & 0 \end{pmatrix} + \frac{1}{HT} \begin{pmatrix} 0 & -\Delta M & 0 & 0 \\ \Delta M & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Delta M_1 \\ 0 & 0 & \Delta M_1 & 0 \end{pmatrix} \quad (5.7)$$

where we recall that $\Delta M_1 \equiv M_+ - M_\phi = M_+^{\text{QED}} + M_+^{\text{CM}} + M_{\text{pl}} - M_\phi$. Here $M_{\text{pl}} = -\omega_{\text{pl}}^2/(2\omega)$ is the term corresponding to plasma effects which is the same for A_+ and A_\times . In order to use perturbation theory, first we must establish which part of the matrix B can be treated as small perturbation.

Suppose first that matrix $B_2(T)$ can be considered as perturbation matrix, namely we can write it as the product of a small temperature independent parameter, ϵ , with temperature depended functions $\tilde{G}(T)$ and $\tilde{G}_1(T)$. This situation would be true when either $|\Delta M(T)| < |\Delta M_1(T)| \ll M_{\phi\gamma}(T)$ or $|\Delta M_1(T)| < |\Delta M(T)| \ll M_{\phi\gamma}(T)$. We will find the corresponding parameter space in Sec. 5. Using the same formalism as we showed in Sec. 4, we get the following solutions (to the first order in ϵ) for U and V components of \tilde{S}_1

¹⁷From now we omit the term generalized for the ‘vectors’ \tilde{S}_1 and \tilde{S}_2 .

$$\begin{aligned}
\left(\frac{T_i}{T}\right)^3 U(T) &= \cos[F_{\phi\gamma}(T)]U_i + \left[\cos[F_{\phi\gamma}(T)] \int_T^{T_i} (G(T') \cos^2[F_{\phi\gamma}(T')] + G_1(T') \sin^2[F_{\phi\gamma}(T')]) dT' + \frac{1}{2} \sin[F_{\phi\gamma}(T)] \right. \\
&\quad \times \int_T^{T_i} \Delta G(T') \sin[2F_{\phi\gamma}(T')] dT' \left. \right] V_i - \sin[F_{\phi\gamma}(T)] S_{4i} - \left[\frac{1}{2} \cos[F_{\phi\gamma}(T)] \int_T^{T_i} \Delta G(T') \sin[2F_{\phi\gamma}(T')] dT' \right. \\
&\quad \left. + \sin[F_{\phi\gamma}(T)] \int_T^{T_i} (G_1(T') \cos^2[F_{\phi\gamma}(T')] + G(T') \sin^2[F_{\phi\gamma}(T')]) dT' \right] S_{5i}, \\
\left(\frac{T_i}{T}\right)^3 V(T) &= - \left[\cos[F_{\phi\gamma}(T)] \int_T^{T_i} (G(T') \cos^2[F_{\phi\gamma}(T')] + G_1(T') \sin^2[F_{\phi\gamma}(T')]) dT' + \frac{1}{2} \sin[F_{\phi\gamma}(T)] \times \right. \\
&\quad \left. \int_T^{T_i} \Delta G(T') \sin[2F_{\phi\gamma}(T')] dT' \right] U_i + \cos[F_{\phi\gamma}(T)] V_i + \left[\frac{1}{2} \cos[F_{\phi\gamma}(T)] \int_T^{T_i} \Delta G(T') \sin[2F_{\phi\gamma}(T')] dT' \right. \\
&\quad \left. + \sin[F_{\phi\gamma}(T)] \int_T^{T_i} (G(T') \sin^2[F_{\phi\gamma}(T')] + G_1(T') \cos^2[F_{\phi\gamma}(T')]) dT' \right] S_{4i} - \sin[F_{\phi\gamma}(T)] S_{5i},
\end{aligned} \tag{5.8}$$

where we have defined $F_{\phi\gamma}(T)$ and $G_1(T)$ respectively as

$$F_{\phi\gamma}(T) \equiv \int_T^{T_i} \frac{M_{\phi\gamma}(T')}{H(T')T'} dT', \quad G_1(T) \equiv \frac{\Delta M_1(T)}{H(T)T},$$

and $\Delta G(T) = G(T) - G_1(T)$. Even though ϵ does not appear explicitly in (5.8), it is implicitly included in $G(T)$ and $G_1(T)$. In (5.8) we show only the solutions for U and V and do not show those for the other components of \tilde{S}_1 since we are not interested in¹⁸.

So far we found the solution for \tilde{S}_1 in the case when elements of the matrix $B_1(T)$ are much bigger in magnitude than elements of $B_2(T)$, where the last matrix has been considered as perturbation matrix. However, for some values of the parameters we have also the situation when $|\Delta M(T)| < M_{\phi\gamma}(T) \ll |\Delta M_1(T)|$. Here we are mostly interested in the case when the pseudoscalar mixing term is bigger than $|\Delta M(T)|$ because the opposite case is fulfilled for uninteresting small values¹⁹ of $g_{\phi\gamma}$. In the case when $|\Delta M(T)| < M_{\phi\gamma}(T) \ll |\Delta M_1(T)|$ it is convenient to move the term $\Delta M(T)$ from matrix $B_2(T)$ to matrix $B_1(T)$. In this case the former matrix has non zero entries only $\Delta M_1(T)$ while the latter matrix has non zero entries $M_{\phi\gamma}(T)$ and $\Delta M(T)$. Now, the matrix $B_2(T)$ can be considered as the leading one while $B_1(T)$ can be considered as perturbation matrix. However, since $M_{\phi\gamma}(T)$ appears now in $B_1(T)$, in order to see the small effects of the pseudoscalar field, it is necessary to look for solution to the second order in ϵ , namely we write $\tilde{S}_1(T) = \tilde{S}_1^{(0)}(T) + \epsilon \tilde{S}_1^{(1)}(T) + \epsilon^2 \tilde{S}_1^{(2)}(T) + \dots$ and insert it in the first equation in (5.6). After collecting all terms and tedious calculations we get the following perturbative solutions for U and V components of \tilde{S}_1 to second order in ϵ

¹⁸In this paper we are only interested in usual Stokes parameters I_γ, Q, U and V since they completely describe the polarization of light. If one is also interested in intensity of pseudoscalar field I_ϕ which is related to S_8 or transition amplitudes of photons into pseudoscalar particles then are needed also expressions for the remaining Stokes parameters S_4, S_5, S_6, S_7, S_8 . We show their solutions in Appendix A.

¹⁹The case $M_{\phi\gamma}(T) \ll |\Delta M(T)|$ essentially means that contribution of pseudoscalar field to the mixing is smaller than the sum of QED and CM effects. Since the last effects are very small in general, see Sec. 4, the case $M_{\phi\gamma}(T) \ll |\Delta M(T)|$ is not of particular interest because it is satisfied for extremely small values of $g_{\phi\gamma}$. If indeed $g_{\phi\gamma}$ is so small, it would be very difficult to experimentally detect pseudoscalar particles, because their signal would be smaller than the QED effect even if perfect laboratory vacuum is achieved.

$$\begin{aligned}
\left(\frac{T_i}{T}\right)^3 U(T) &= \left(1 - \int_T^{T_i} G(T')dT' \int_{T'}^{T_i} G(T'')dT'' - \int_T^{T_i} \cos[\mathcal{G}_1(T')]G_{\phi\gamma}(T')dT' \int_{T'}^{T_i} \cos[\mathcal{G}_1(T'')]G_{\phi\gamma}(T'')dT'' \right. \\
&\quad \left. - \int_T^{T_i} \sin[\mathcal{G}_1(T')]G_{\phi\gamma}(T')dT' \int_{T'}^{T_i} \sin[\mathcal{G}_1(T'')]G_{\phi\gamma}(T'')dT'' \right) U_i + \left(\int_T^{T_i} G(T')dT' + \int_T^{T_i} \cos[\mathcal{G}_1(T')] \right. \\
&\quad \left. \times G_{\phi\gamma}(T')dT' \int_{T'}^{T_i} \sin[\mathcal{G}_1(T'')]G_{\phi\gamma}(T'')dT'' - \int_T^{T_i} \sin[\mathcal{G}_1(T')]G_{\phi\gamma}(T')dT' \int_{T'}^{T_i} \cos[\mathcal{G}_1(T'')]G_{\phi\gamma}(T'')dT'' \right) V_i \\
&\quad \left(\int_T^{T_i} G(T')dT' \int_{T'}^{T_i} \sin[\mathcal{G}_1(T'')]G_{\phi\gamma}(T'')dT'' - \int_T^{T_i} \cos[\mathcal{G}_1(T')]G_{\phi\gamma}(T')dT' \right) S_{4i} - \\
&\quad \left(\int_T^{T_i} G(T')dT' \int_{T'}^{T_i} \cos[\mathcal{G}_1(T'')]G_{\phi\gamma}(T'')dT'' + \int_T^{T_i} \sin[\mathcal{G}_1(T')]G_{\phi\gamma}(T')dT' \right) S_{5i}, \\
\left(\frac{T_i}{T}\right)^3 V(T) &= - \left(\int_T^{T_i} G(T')dT' + \int_T^{T_i} \cos[\mathcal{G}_1(T')]G_{\phi\gamma}(T')dT' \int_{T'}^{T_i} \sin[\mathcal{G}_1(T'')]G_{\phi\gamma}(T'')dT'' - \int_T^{T_i} \sin[\mathcal{G}_1(T')]G_{\phi\gamma}(T')dT' \right. \\
&\quad \left. \times \int_{T'}^{T_i} \cos[\mathcal{G}_1(T'')]G_{\phi\gamma}(T'')dT'' \right) U_i + \left(1 - \int_T^{T_i} G(T')dT' \int_{T'}^{T_i} G(T'')dT'' - \int_T^{T_i} \cos[\mathcal{G}_1(T')]G_{\phi\gamma}(T')dT' \right. \\
&\quad \left. \times \int_{T'}^{T_i} \cos[\mathcal{G}_1(T'')]G_{\phi\gamma}(T'')dT'' - \int_T^{T_i} \sin[\mathcal{G}_1(T')]G_{\phi\gamma}(T')dT' \int_{T'}^{T_i} \sin[\mathcal{G}_1(T'')]G_{\phi\gamma}(T'')dT'' \right) V_i \\
&\quad + \left(\int_T^{T_i} G(T')dT' \int_{T'}^{T_i} \cos[\mathcal{G}_1(T'')]G_{\phi\gamma}(T'')dT'' + \int_T^{T_i} \sin[\mathcal{G}_1(T')]G_{\phi\gamma}(T')dT' \right) S_{4i} \\
&\quad + \left(\int_T^{T_i} G(T')dT' \int_{T'}^{T_i} \sin[\mathcal{G}_1(T'')]G_{\phi\gamma}(T'')dT'' - \int_T^{T_i} \cos[\mathcal{G}_1(T')]G_{\phi\gamma}(T')dT' \right) S_{5i}, \quad (5.9)
\end{aligned}$$

where we have defined $\mathcal{G}_1(T)$ and $G_{\phi\gamma}(T)$ respectively as

$$\mathcal{G}_1(T) = \int_T^{T_i} G_1(T')dT', \quad G_{\phi\gamma}(T) = M_{\phi\gamma}/(HT).$$

5.3 Solution of second reduced Stokes vector \tilde{S}_2

Now we focus on the solution of second reduced Stokes vector \tilde{S}_2 which is the only one left. Even in this case we look for approximate solution and use perturbation theory in analogous way with the previous section. It is convenient to split the matrix $C(T)$ which enters in the second equation in (5.6) in the following order, $C(T) = C_1(T) + C_2(T)$

$$C_1 + C_2 = \frac{1}{HT} \begin{pmatrix} 0 & 0 & M_{\phi\gamma} & 0 & 0 \\ 0 & 0 & -M_{\phi\gamma} & 0 & 0 \\ 0 & M_{\phi\gamma} & 0 & 0 & -\sqrt{3}M_{\phi\gamma} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3}M_{\phi\gamma} & 0 & 0 \end{pmatrix} + \frac{1}{HT} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Delta M_2 & 0 \\ 0 & 0 & \Delta M_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (5.10)$$

where $\Delta M_2 = M_{\times} - M_{\phi} = M_{\times}^{\text{QED}} + M_{\times}^{\text{CM}} + M_{\text{pl}} - M_{\phi}$. At this point we must establish which matrix in (5.10) can be considered as perturbation matrix. This can be done by comparing the elements of $C_1(T)$ with C_2 . In the case when $|\Delta M_2(T)| \ll M_{\phi\gamma}(T)$, the matrix $C_2(T)$ can be considered as perturbation matrix and vice-versa in the case $|\Delta M_2(T)| \gg M_{\phi\gamma}(T)$.

In case when $|\Delta M_2(T)| \ll M_{\phi\gamma}(T)$, we get the following solutions²⁰ for I_{γ} and Q components of \tilde{S}_2

²⁰In case when the term $M_{\phi\gamma}(T)$ is much bigger than $|\Delta M_2(T)|$, it is not necessary to go beyond the first order in ϵ in perturbation theory, since the effects of the pseudoscalar field are already evident to first order in ϵ .

to first order in ϵ

$$\begin{aligned}
\left(\frac{T_i}{T}\right)^3 I_\gamma(T) &= I_\gamma(T_i) + \frac{1}{2} \sin^2[F_{\phi\gamma}(T)] Q_i - \frac{1}{2} \sin[2F_{\phi\gamma}(T)] S_{6i} + \left[\frac{1}{2} \cos[2F_{\phi\gamma}(T)] \int_T^{T_i} G_2(T') \sin[2F_{\phi\gamma}(T')] dT' \right. \\
&\quad \left. - \frac{1}{2} \sin[2F_{\phi\gamma}(T)] \int_T^{T_i} G_2(T') \cos[2F_{\phi\gamma}(T')] dT' \right] S_{7i} - \frac{\sqrt{3}}{2} \sin^2[F_{\phi\gamma}(T)] S_{8i}, \\
\left(\frac{T_i}{T}\right)^3 Q(T) &= \frac{1}{4} (3 + \cos[2F_{\phi\gamma}(T)]) Q_i + \frac{1}{2} \sin[2F_{\phi\gamma}(T)] S_{6i} + \left[-\frac{1}{2} \cos[2F_{\phi\gamma}(T)] \int_T^{T_i} G_2(T') \sin[2F_{\phi\gamma}(T')] dT' \right. \\
&\quad \left. + \frac{1}{2} \sin[2F_{\phi\gamma}(T)] \int_T^{T_i} G_2(T') \cos[2F_{\phi\gamma}(T')] dT' \right] S_{7i} + \frac{\sqrt{3}}{2} \sin^2[F_{\phi\gamma}(T)] S_{8i}, \tag{5.11}
\end{aligned}$$

where we defined $G_2(T) = \epsilon \tilde{G}_2(T) = \Delta M_2(T)/(HT)$. As in the previous section ϵ does not explicitly appear in (5.11) but is implicitly included in $G_2(T)$.

The case $|\Delta M_2(T)| \gg M_{\phi\gamma}(T)$, needs a special treatment because the term corresponding to the pseudoscalar field is subdominant. In order to explore the vast region of pseudoscalar particles parameter space, it is necessary to look for solution of $\tilde{S}_2(T)$ up to second order in ϵ . Therefore, we expand the second reduced Stokes vector as $\tilde{S}_2(T) = \tilde{S}_2^{(0)}(T) + \epsilon \tilde{S}_2^{(1)}(T) + \epsilon^2 \tilde{S}_2^{(2)}(T) + \dots$ and insert it in the second equation in (5.6). Collecting all terms, we get the following perturbative solutions for I_γ and Q components of \tilde{S}_2 to second order²¹ in ϵ :

$$\begin{aligned}
\left(\frac{T_i}{T}\right)^3 I_\gamma(T) &= I_\gamma(T_i) + \left(\int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \cos[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' + \right. \\
&\quad \left. \int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \sin[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' \right) Q_i - \left(\int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \right) S_{6i} \\
&\quad - \left(\int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \right) S_{7i} - \sqrt{3} \left(\int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \right. \\
&\quad \left. \times \int_{T'}^{T_i} \cos[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' + \int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \sin[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' \right) S_{8i}, \\
\left(\frac{T_i}{T}\right)^3 Q(T) &= \left(1 - \int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \cos[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' - \right. \\
&\quad \left. \int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \sin[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' \right) Q_i + \left(\int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \right) S_{6i} \\
&\quad + \left(\int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \right) S_{7i} + \sqrt{3} \left(\int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \right. \\
&\quad \left. \times \int_{T'}^{T_i} \cos[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' + \int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \sin[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' \right) S_{8i}, \tag{5.12}
\end{aligned}$$

where $\mathcal{G}_2(T) = \int_T^{T_i} G_2(T') dT'$.

6 Pseudoscalar particle production and generation of CMB polarization

In Sec. 5 we solved the equation of motion for the reduced Stokes vectors in case of perpendicular propagation with respect to the external magnetic field \mathbf{B}_e . This particular configuration, allowed us to solve the equations of motion by using perturbation theory. In this section we focus on the impact

²¹As we did above, the small factor ϵ in this case is implicitly included in $M_{\phi\gamma}(T)$.

of pseudoscalar particle production in generation of CMB circular polarization. In what follows, we concentrate only in generation of CMB polarization after the decoupling epoch and estimate the degree of circular at present epoch. In order to make our treatment as simple as possible, it would be very convenient to study the case when $M_{\phi\gamma}(T)$ is the dominant term separately from the case when it is subdominant with respect to the other terms $\Delta M_1(T)$ and $\Delta M_2(T)$.

6.1 Dominant pseudoscalar contribution: $M_{\phi\gamma} \gg |\Delta M|, |\Delta M_1|, |\Delta M_2|$

In the case when the term $M_{\phi\gamma}(T)$ is bigger than other terms in matrices $B(T)$ and $C(T)$, which essentially corresponds to the strong mixing case, the solutions of equations of motion for the usual Stokes parameters are given by (5.8) and (5.11). Let us concentrate at the post decoupling epoch and assume that at $T = T_i$ the CMB is very weakly polarized due to Thomson scattering and consider the generation and evolution of polarization for $T \leq T_i$. In what follows, we assume that the cosmological plasma is not populated by other relic pseudoscalar particles at $T \leq T_i$, the CMB is not circularly polarized at $T = T_i$ and significant pseudoscalar particle production starts at $T = T_i$ in already existing cosmic magnetic field. In this case we have, $V_i = S_{4i} = S_{5i} = S_{6i} = S_{7i} = 0$ and conservation of particle number gives $I_\gamma(T_i) = \sqrt{3}S_{8i}$. We use these values as initial conditions in expressions (5.8) and (5.11). In this section we are not interested in the evolution of other Stokes parameters and will not be considered. Before moving to the study of CMB polarization, it is first necessary to find the pseudoscalar parameter space which satisfy the conditions $M_{\phi\gamma}(T) \gg |\Delta M(T)| > |\Delta M_1(T)|$ or $M_{\phi\gamma}(T) \gg |\Delta M_1(T)| > |\Delta M(T)|$ and $M_{\phi\gamma}(T) \gg |\Delta M_2(T)|$.

The cases $M_{\phi\gamma}(T) \gg |\Delta M(T)| > |\Delta M_1(T)|$ or $M_{\phi\gamma}(T) \gg |\Delta M_1(T)| > |\Delta M(T)|$ and $M_{\phi\gamma}(T) \gg |\Delta M_2(T)|$ can be solved in principle exactly, but it would be quite involved to study all possibilities of these inequality equations. However, in most practical cases it is sufficient only to calculate the leading terms in each member of the inequalities. In order to do so, let us recall that $\Delta M(T) = \Delta M_{\text{QED}}(T) + \Delta M_{\text{CM}}(T)$ and $\Delta M_1(T) = M_+^{\text{QED}}(T) + M_+^{\text{CM}}(T) + M_{\text{pl}}(T) - M_\phi(T)$. For the QED term we have essentially $|\Delta M_{\text{QED}}(T)| \sim M_+^{\text{QED}}(T) \sim M_\times^{\text{QED}}(T)$ and for the CM term, $\Delta M_{\text{CM}}(T) = -M_\times^{\text{CM}}(T)$ with $M_+^{\text{CM}} = 0$ since the transverse part of external magnetic field has no y component by convention. Since the plasma term is in general several orders of magnitude much bigger than QED and CM terms in ΔM_1 , in the parameter space that we are interested in at the post decoupling epoch, we have essentially $\Delta M_1 \simeq M_{\text{pl}} - M_\phi$ where here we are also assuming that $|M_\phi|$ is bigger than QED and CM terms. Therefore, it remains to confront the term M_ϕ with the plasma term M_{pl} , where the former depends on the pseudoscalar mass m_ϕ . Therefore we have either $|M_\phi(T)| > |M_{\text{pl}}(T)|$ or $|M_\phi(T)| < |M_{\text{pl}}(T)|$. Consequently, depending on the pseudoscalar mass, we have respectively either $|\Delta M_1(T)| \simeq |M_\phi(T)|$ or $|\Delta M_1(T)| \simeq |M_{\text{pl}}(T)|$. Being the plasma term much bigger than QED and CM terms, the previous conditions imply that in most cases we have $|\Delta M_1(T)| > |\Delta M(T)|$. Based on the same arguments one can easily show that also $\Delta M_2 \simeq M_{\text{pl}} - M_\phi$.

The conditions $M_{\phi\gamma} \gg |\Delta M_{1,2}|$, in the case when $|M_\phi| > |M_{\text{pl}}|$ are satisfied for all $T_0 \leq T \leq T_i$ if²²

$$2 \times 10^{-10} \text{ eV} < m_\phi, \quad g_{\phi\gamma} \gg 1.24 \times 10^{25} \left(\frac{\text{Hz}}{\nu_0} \right) \left(\frac{m_\phi}{\text{eV}} \right)^2 \left(\frac{\text{G}}{B_{e0}} \right) \text{ GeV}^{-1}, \quad (6.1)$$

where we used the fact that for $|M_\phi| > |M_{\text{pl}}|$, $M_{\phi\gamma} \gg |M_\phi|$. In the other case, namely $|M_\phi| < |M_{\text{pl}}|$ and $M_{\phi\gamma} \gg |\Delta M_{1,2}| \simeq |M_{\text{pl}}|$ are satisfied for all $T_0 \leq T \leq T_i$ if

$$m_\phi < 1.6 \times 10^{-14} \text{ eV}, \quad g_{\phi\gamma} \gg 3.22 \times 10^{-3} \left(\frac{\text{Hz}}{\nu_0} \right) \left(\frac{\text{G}}{B_{e0}} \right) \text{ GeV}^{-1}. \quad (6.2)$$

²²It is important to stress that since we are working with perturbation theory, the conditions $M_{\phi\gamma} \gg |\Delta M_{1,2}|$ for $|M_\phi| > |M_{\text{pl}}|$ must be satisfied in the whole interval $T_0 \leq T \leq T_i$. They are respectively satisfied when their temperature dependent terms $(T_0/T)^3$ and $\sqrt{X_e(T)}(T/T_0)^{3/2}$ are maximum. On the other hand, the conditions $M_{\phi\gamma} \gg |\Delta M_{1,2}|$ for $|M_{\text{pl}}| > |M_\phi|$ must be satisfied in the whole interval $T_0 \leq T \leq T_i$ when the temperature dependent terms $X_e(T)$ and $\sqrt{X_e(T)}(T/T_0)^{3/2}$ are respectively maximum and minimum.

As last condition, it remains to find the validity interval of the resonant case which occurs for transverse magnetic field when $M_\times(T) = M_\phi(T)$. In this case we have that $\Delta M_2(T) = 0$ which implies $\Delta M_1(T) = \Delta M(T)$. When this condition is met, the equations for the vectors \tilde{S}_1 and \tilde{S}_2 are solved exactly. The expressions for the parameters I_γ and Q are still given by (5.11) where $G_2(T) = 0$, while the expressions for the parameters U and V are different now and in the case of initial conditions $V_i = S_{4i} = S_{5i} = 0$ are given by $(T_i/T)^3 U(T) = \cos[\mathcal{G}(T)] \cos[F_{\phi\gamma}(T)] U_i$ and $(T_i/T)^3 V(T) = -\sin[\mathcal{G}(T)] \cos[F_{\phi\gamma}(T)] U_i$.

Having discussed the limits of validity of perturbation theory, let us now concentrate on the generation of circular polarization. The expression for the degree of circular polarization, in the case of resonant mixing, is given by the ration $|V|/I_\gamma$ which reads

$$P_C(T_0) = \frac{|-\sin[\mathcal{G}(T_0)] \cos[F_{\phi\gamma}(T_0)] U_i|}{1 - (1/2) \sin^2[F_{\phi\gamma}(T_0)](1 - Q_i)}. \quad (6.3)$$

From (6.3) we may observe that the degree of circular polarization today can be written as $P_C(T_0) = f(F_{\phi\gamma}) P_C^{\text{QED+CM}}(T_0)$ where $f(F_{\phi\gamma})$ is a trigonometric function with absolute value of $0 \leq |f(F_{\phi\gamma})| \leq 1$ and $P_C^{\text{QED+CM}}$ is the degree of circular polarization due to QED and CM effects. As we saw in Sec. 4 for the case of $\Phi = \pi/2$, the degree of circular polarization due to QED and CM effects is much smaller than the present limit on $P_C(T_0)$ set by MIPOL experiment [25], $P_C(T_0) < 7 \times 10^{-5}$. In principle one can use (6.3) together with MIPOL upper limit to constrain $g_{\phi\gamma}$ in the resonant mixing. However, due to the function $f(F_{\phi\gamma})$, the contribution of resonant photon-pseudoscalar mixing to total circular polarization is at maximum ≤ 1 times the contributions of QED and CM effects. This implies that the upper limit of MIPOL would be satisfied for *any* value of $g_{\phi\gamma}$ and for $m_\phi(T) \simeq 1.6 \times 10^{-14} \sqrt{X_e(T)} (T/T_0)^{3/2}$ eV. The last expression comes from the fact that in resonance $\Delta M_2(T) = 0$ which is satisfied when $M_\phi = M_{\text{pl}} + M_\times^{\text{QED}} + M_\times^{\text{CM}}$, where the pseudoscalar particle acquires a temperature (or time) depended mass due to resonant mixing with the photon. The marginal contribution to P_C due to pseudoscalar field is in agreement with results found in Sec. 2 in the case of transverse magnetic field and resonant photon-pseudoscalar mixing, namely there is not induced additional phase shift due to the pseudoscalar field.

Now let us focus in the case of strong mixing but not maximal mixing. Again we are interested in the degree of circular polarization at the present time $P_C(T_0)$. The expression for the parameter V in the strong mixing is given in (5.8). Assuming the CMB linearly polarized at decoupling, namely using our initial conditions as described above, only the first term in V proportional to U_i is present in (5.8). There are three different types of integrals present in the first term proportional to U_i in (5.8) which can be calculated analytically. Before evaluating these integrals, it is convenient to extract the temperature dependence on each term inside integrals. Therefore, we can write for example $G(T) = \mathcal{C}_1 T^{5/2}$ where $\mathcal{C}_1 = -8.12 \times 10^{-14} \nu_0 B_{e0}^2 T_0^{-5/2}$ ($\text{K}^{-7/2}$) where ν_0 is measured in units of Hz, B_{e0} in units of G and T_0 is measured in units of Kelvin. We can do the same thing for $F_{\phi\gamma}$ and write $F_{\phi\gamma}(T) = \mathcal{C}_2 (\sqrt{T_i} - \sqrt{T})$ where $\mathcal{C}_2 = 9.74 \times 10^{21} T_0^{1/2} g_{\phi\gamma} B_{e0}$ ($\text{K}^{-1/2}$) with $g_{\phi\gamma}$ measured in units of GeV^{-1} .

By integrating, the first integral proportional to U_i in (5.8) is given by

$$\begin{aligned} \int_T^{T_i} G(T') \cos^2[F_{\phi\gamma}(T')] dT' = & 0.0178(\mathcal{C}_1/\mathcal{C}_2^7) \left[-42\mathcal{C}_2\sqrt{T} (2\mathcal{C}_2^4 T^2 - 10\mathcal{C}_2^2 T + 15) \cos \left[\mathcal{C}_2 (108.99 - 2\sqrt{T}) \right] \right. \\ & - 8\mathcal{C}_2^7 T^{7/2} + 1.14 \times 10^{13} \mathcal{C}_2^7 + 4.03 \times 10^{10} \mathcal{C}_2^5 - 6.79 \times 10^7 \mathcal{C}_2^3 \\ & \left. + 7(4\mathcal{C}_2^6 T^3 - 30\mathcal{C}_2^4 T^2 + 90\mathcal{C}_2^2 T - 45) \sin \left(108.995\mathcal{C}_2 - 2\mathcal{C}_2\sqrt{T} \right) + 34333.6\mathcal{C}_2 \right], \end{aligned} \quad (6.4)$$

where we took $T_i = 2970$ K. The integral in (6.4) can be further simplified by observing that \mathcal{C}_2 is a very large number. Indeed, since we are in the strong mixing regime and since $\mathcal{C}_2 \propto g_{\phi\gamma}$, we have from either (6.1) or (6.2) that for a canonical value of $B_{e0} \sim \text{nG}$, $\mathcal{C}_2 \gg 1$ if either (6.1) or (6.2) applies. In this case it is only the third term in (6.4) which dominates over the other terms for $T < T_i$ and the integral can

be approximated to high accuracy as

$$\int_{T_0}^{T_i} G(T') \cos^2[F_{\phi\gamma}(T')] dT' \simeq 2 \times 10^{11} \mathcal{C}_1 \quad (\text{K}^{7/2}). \quad (6.5)$$

Consider now the second integral which appears in the term proportional to U_i in (5.8). It contains under the sign of integral the product of $G_1(T)$ with $\sin^2[F_{\phi\gamma}(T)]$. The function $G_1(T)$ is proportional to $\Delta M_1(T)$ where the latter can be either proportional to M_{pl} if $|M_{\text{pl}}| > |M_\phi|$ or to M_ϕ if $|M_{\text{pl}}| < |M_\phi|$. In the case when $|M_{\text{pl}}| > |M_\phi|$ we can write $G_1(T) = \mathcal{C}_3 T^{-1/2}$ where $\mathcal{C}_3 = -1.56 \times 10^{19} T_0^{1/2} \nu_0^{-1} \bar{X}_e$ ($\text{K}^{-1/2}$) and for simplicity we took the average value of $X_e(T)$ at post decoupling epoch $\bar{X}_e \simeq 0.023$. The analytic expression for the second integral is given by

$$\int_T^{T_i} G_1(T') \sin^2[F_{\phi\gamma}(T')] dT' = \mathcal{C}_3 \left(54.49 - \sqrt{T} - \frac{\sin[108.99\mathcal{C}_2 - 2\mathcal{C}_2 \sqrt{T}]}{2\mathcal{C}_2} \right), \quad (6.6)$$

where in the case of $\mathcal{C}_2 \gg 1$, the last term in (6.7) can be safely neglected.

The last integral which appears in the term proportional to U_i in (5.8), has under the integral sign the product of $\Delta G(T)$ with $\sin[2F_{\phi\gamma}(T)]$. The calculation of this integral can be simplified by noting that $\Delta G(T) = -G_2(T)$. Since we are in the strong mixing but not maximal mixing, we have essentially that $|\Delta M_1| \simeq |\Delta M_2|$ and consequently we can approximate $|G_2| \simeq |G_1|$. Therefore, we get

$$\int_T^{T_i} \Delta G(T') \sin[2F_{\phi\gamma}(T')] dT' \simeq - \int_T^{T_i} G_2(T') \sin[2F_{\phi\gamma}(T')] dT' = 2(\mathcal{C}_3/\mathcal{C}_2) \sin^2[54.49\mathcal{C}_2 - \mathcal{C}_2 \sqrt{T}], \quad (6.7)$$

where we considered the regime where $|M_{\text{pl}}| > |M_\phi|$. Using expressions (6.5)-(6.7), we get the following expression for V at $T = T_0$

$$(T_i/T_0)^3 V(T_0) = \left(-\cos[F_{\phi\gamma}(T_0)] [2 \times 10^{11} \mathcal{C}_1 + 52.84 \mathcal{C}_3] + \frac{1}{2} \sin[F_{\phi\gamma}(T_0)] (\mathcal{C}_3/\mathcal{C}_2) [1 - \cos[105.694 \mathcal{C}_2]] \right) U_i, \quad (6.8)$$

where we dropped the measurement units in V_0 for convenience reason. The expression for V_0 can be further simplified by noting that in most cases we have $|\mathcal{C}_1| \ll |\mathcal{C}_3| \ll \mathcal{C}_2$. Since the values of trigonometric functions are bounded between zero and one, we can approximate (6.8) to higher accuracy with

$$(T_i/T_0)^3 V(T_0) \simeq -52.84 \cos[F_{\phi\gamma}(T_0)] \mathcal{C}_3 U_i. \quad (6.9)$$

Having found the expression for V_0 now we can easily find the expression for the degree of circular polarization in the strong mixing case for $|M_{\text{pl}}| > |M_\phi|$. Indeed, using (5.11) and neglecting the term proportional to Q_i in I_γ , we get

$$P_C(T_0) \simeq \frac{|-52.84 \cos[F_{\phi\gamma}(T_0)] \mathcal{C}_3 U_i|}{1 - (1/2) \sin^2[F_{\phi\gamma}(T_0)]}. \quad (6.10)$$

The degree of circular polarization in the strong mixing regime depends explicitly on frequency ν_0 through \mathcal{C}_3 . Considering for example that $0 < P_C(T_0) < 7 \times 10^{-5}$ (MIPOL) for $\nu_0 = 33$ GHz, the solutions²³ of the trigonometric inequation (6.10) in terms of $g_{\phi\gamma} B_{e0}$ jointly with condition (6.2) are given by

$$5.38 \times 10^{-25} (4n\pi - \pi) < g_{\phi\gamma} B_{e0} < 2.35 \times 10^{-24} (n\pi - \pi/4) \quad \text{or} \\ 2.35 \times 10^{-24} (n\pi + \pi/4) < g_{\phi\gamma} B_{e0} < 5.88 \times 10^{-25} (4n\pi + \pi), \quad (6.11)$$

²³The solutions of trigonometric equations or inequations which we present in this and next sections are in general approximate. In general, these solutions depends on an integer $n \in \mathbf{Z}$ which is a very large number. In all cases that we present below we approximate n with a real number, such for example the integer $n = 13133911539$ with $n = 1.31 \times 10^{10}$. For precise numerical solutions one must keep the exact value of n and other numerical expressions.

where $n \gg 1.31 \times 10^{10}$ with $n \in \mathbf{Z}$, B_{e0} is measured in units of Gauss, $g_{\phi\gamma}$ in units of GeV^{-1} and we took $U_i \simeq -Q_i$ with $Q_i \simeq 10^{-6}$ at decoupling. Expression (6.11), for fixed value of B_{e0} , gives infinite values of $g_{\phi\gamma}$ namely a degeneracy, that satisfy MIPOL upper limit on $P_C(T_0)$. Similar solutions, namely degenerate values of $g_{\phi\gamma}$, are also found in case when $|M_{\text{pl}}| < |M_\phi|$. These multiple solutions should not surprise since we are dealing with trigonometric inequation which in general has multiple solutions. In principle one can further reduce the range of values of $g_{\phi\gamma}B_{e0}$ by using other constraints obtained by complementary methods.

6.2 Subdominant pseudoscalar contribution: $M_{\phi\gamma} \ll |\Delta M_1|, |\Delta M_2|$

Now let us turn our attention to the case of weak mixing which occurs when $M_{\phi\gamma}(T) \ll |\Delta M_{1,2}(T)|$. In this case the expressions for the Stokes parameters I_γ and Q are given by (5.12) while for U and V are given by (5.9). Also here we consider the following initial conditions to be applied to Stokes parameters, $V_i = S_{4i} = S_{5i} = S_{6i} = S_{7i} = 0$ at $T = T_i$.

It is important at this stage to find the pseudoscalar parameter space that satisfy the condition of weak mixing. As we saw in the case of dominant pseudoscalar contribution, we have $|\Delta M(T)| < |\Delta M_1(T)|$ if we are not in the maximal mixing regime. Since this is obviously the case, it remains to find the parameter space only for $M_{\phi\gamma}(T) \ll |\Delta M_{1,2}(T)|$ where in most practical cases we have $|\Delta M_1| \simeq |\Delta M_2|$ for $|M_\phi| \neq |M_{\text{pl}}|$. Using the same arguments as we did in the previous section, we find that conditions $|M_\phi| > |M_{\text{pl}}|$ and $M_{\phi\gamma} \ll |\Delta M_{1,2}|$ are satisfied for all T at the post decoupling if

$$2 \times 10^{-10} \text{ eV} < m_\phi, \quad g_{\phi\gamma} \ll 9.57 \times 10^{15} \left(\frac{\text{Hz}}{\nu_0} \right)^2 \left(\frac{\text{G}}{B_{e0}} \right) \text{ GeV}^{-1}, \quad (6.12)$$

where for $|M_\phi| > |M_{\text{pl}}|$, $M_{\phi\gamma} \ll |M_\phi|$. In the case when $|M_\phi| < |M_{\text{pl}}|$ we have $M_{\phi\gamma} \ll |\Delta M_{1,2}| \simeq |M_{\text{pl}}|$ which are satisfied for all T at post decoupling epoch if

$$m_\phi < 1.6 \times 10^{-14} \text{ eV}, \quad g_{\phi\gamma} \ll 3.22 \times 10^{-3} \bar{X}_e \left(\frac{\text{Hz}}{\nu_0} \right) \left(\frac{\text{G}}{B_{e0}} \right) \text{ GeV}^{-1}, \quad (6.13)$$

where \bar{X}_e is the average value of $X_e(T)$ at the post decoupling epoch. The reason of having chosen the average value will be clear below.

Now that we have established limits of validity for I_γ and Q ($M_{\phi\gamma}(T) \ll |\Delta M_2(T)|$) and for U and V ($M_{\phi\gamma}(T) \ll |\Delta M_1(T)|$) in the weak mixing, we can focus on the generation of CMB circular polarization. In the Stokes parameters I_γ and Q do appear trigonometric functions that have as argument $\mathcal{G}_2(T)$ while in U and V have as argument $\mathcal{G}_1(T)$. Since these functions are given respectively by the integral of $\Delta M_2(T)$ and $\Delta M_1(T)$, it may be convenient to separate if either the plasma term M_{pl} or the mass term M_ϕ dominates in $\Delta M_{1,2}$. As already mentioned, the QED and CM terms are much smaller than the plasma term. Consider first the case when the plasma term M_{pl} dominates in $\Delta M_{1,2}$. Considering only the matter contribution to the Hubble parameter $H(T)$, we get

$$\mathcal{G}_1(T) \simeq \mathcal{G}_2(T) = -1.56 \times 10^{19} \sqrt{T_0} (\text{Hz}/\nu_0) \int_T^{T_i} T'^{-1/2} X_e(T') dT' \quad (\text{K}^{-1}).$$

We may note that $\mathcal{G}_{1,2}$ is given as the integral of the inverse square root of temperature times the ionization fraction X_e . As already discussed in Sec. 4, there is not an analytic function for X_e which satisfies a complicated differential equation. In Sec. 4 we were able to find semi-analytic solutions for integrals involving trigonometric functions which have as argument integrals of X_e . For most practical cases, the argument of those trigonometric functions was much smaller than unity, but here we may note that $\mathcal{G}_{1,2}$ is never less than unity for realistic values of ν_0 . So, in this section we cannot approximate the cosine or sine of $\mathcal{G}_{1,2}$ with unity or $\mathcal{G}_{1,2}$ to first order.

It is desirable to have analytic or at least semi-analytic expression for the degree of circular polarization as we did in Sec. 4. Since there is no known analytic expression for X_e , it is convenient to replace

it with its average value in $\mathcal{G}_{1,2}$ at post decoupling epoch, namely $\bar{X}_e \simeq 0.023$. Putting \bar{X}_e into $\mathcal{G}_{1,2}$ we obtain $\mathcal{G}_{1,2}(T) \simeq -1.19 \times 10^{18} (\text{Hz}/\nu_0)(\sqrt{T_i} - \sqrt{T}) (\text{K}^{-1/2})$. Now with $\mathcal{G}_2(T)$ given, we can calculate the integrals which appear in I_γ in expression (5.12). By integrating, we obtain

$$\begin{aligned}\mathcal{I}_c &= \int_{T_0}^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \cos[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' = \tilde{a}^{-2} b (1 - \cos[105.69 \tilde{a}]), \\ \mathcal{I}_s &= \int_{T_0}^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \sin[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' = \tilde{a}^{-2} b (3 - 4 \cos[52.84 \tilde{a}] + \cos[105.96 \tilde{a}]),\end{aligned}\quad (6.14)$$

where we have defined $\tilde{a} \equiv 1.19 \times 10^{18} (\text{Hz}/\nu_0)$ and $b \equiv 6.47 \times 10^{43} (g_{\phi\gamma}/\text{GeV}^{-1})^2 (B_{e0}/\text{G})^2$. Defining $y = \mathcal{I}_c + \mathcal{I}_s$, we get the following expression for the intensity at present

$$(T_i/T_0)^3 I_\gamma(T_0) = 1 - y + y Q_i, \quad (6.15)$$

where $y \equiv 4 \tilde{a}^{-2} b (1 - \cos[52.84 \tilde{a}])$ and $I_i = 1$.

Let us concentrate on V parameter in (5.9) and on the first term proportional to U_i , since other terms are absent with our choice of initial conditions. We may note the first term within parenthesis which corresponds to the QED and CM effects while other terms correspond to mixing of pseudoscalar term with QED and CM terms. The first thing to point out, is that the term corresponding to the QED and CM effects is smaller than other terms because we are in the situation when $|\Delta M|$ is smaller than $M_{\phi\gamma}$. The second thing to note is that appear double integrals which involve sine and cosine functions in the same integral. Let \mathcal{I}_{cs} be the double integral in the order cosine and sine functions and \mathcal{I}_{sc} be the integral for the opposite order. In case when the plasma term M_{pl} dominates M_ϕ in \mathcal{G}_1 , is possible to find analytic expressions for \mathcal{I}_{cs} and \mathcal{I}_{sc} which are respectively given by

$$\begin{aligned}\mathcal{I}_{cs} &= \int_{T_0}^{T_i} \cos[\mathcal{G}_1(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \sin[\mathcal{G}_1(T'')] G_{\phi\gamma}(T'') dT'' = 105.69 \tilde{a}^{-1} b + \\ &\quad \tilde{a}^{-2} b (\sin[105.69 \tilde{a}] - 4 \sin[52.84 \tilde{a}]), \\ \mathcal{I}_{sc} &= \int_{T_0}^{T_i} \sin[\mathcal{G}_1(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \cos[\mathcal{G}_1(T'')] G_{\phi\gamma}(T'') dT'' = -105.69 \tilde{a}^{-1} b + \tilde{a}^{-2} b \sin[105.69 \tilde{a}].\end{aligned}$$

Now putting expressions for \mathcal{I}_{cs} and \mathcal{I}_{sc} in the first term in $V(T)$, we get

$$(T_i/T_0)^3 V(T_0) = -(\mathcal{G}(T_0) + 211.38 \tilde{a}^{-1} b - 4 \tilde{a}^{-2} b \sin[52.84 \tilde{a}]) U_i. \quad (6.16)$$

Using (6.16) and (6.15) together with expression for y , we get the following expression for the degree of circular polarization $|V|/I_\gamma$ at $T = T_0$ to second order in perturbation theory

$$P_C(T_0) = \frac{|-(\mathcal{G}(T_0) + 211.38 \tilde{a}^{-1} b - 4 \tilde{a}^{-2} b \sin[52.84 \tilde{a}]) U_i|}{1 - 4b \tilde{a}^{-2} (1 - \cos[52.84 \tilde{a}]) (1 - Q_i)}. \quad (6.17)$$

An important thing to note is that photon intensity must decrease and never increase in the case of pseudoscalar particle production. This happens because in our case we are not assuming photon injection in the medium by some external source that would eventually increase the photon intensity. Since I_γ must decrease or at least remain constant, this implies that the quantity $-y + y Q_i$ in (6.15) must be negative. Indeed, this is true and can be easily verified by evaluating the second and third terms in (6.15) for a given frequency. All told, is a necessary condition but not sufficient. We must also require that $1 - y + y Q_i > 0$ since the intensity is a positive quantity. If we consider for example the working frequency of MIPOL experiment, $\nu_0 = 33 \text{ GHz}$, for $1 - y + y Q_i > 0$ to be satisfied²⁴ we

²⁴The function $1 - \cos[52.84 \tilde{a}]$ in y is extremely fast oscillating one and for correct evaluation for a given frequency is better to keep several digits. In this work for $\nu_0 = 33 \text{ GHz}$ we used the value of 1.82.

must have $|g_{\phi\gamma}| < 1.66 \times 10^{-15} (G/B_{e0}) \text{ GeV}^{-1}$. Another consideration to be made is related with the parameter V in (6.16). As we can see in (6.16) the first term on the r. h. s. corresponds to the QED and CM effects while the second and third corresponds essentially to mixed terms. However, since perturbative expansion to second order does not reveal the asymptotic behaviour of the series and because we expect that pseudoscalar contribution to V to be small or $(T_i/T_0)^3 V(T_0) \lesssim U_i$, we require the additional condition that the dominant term $211.38 \tilde{a}^{-1} b \lesssim 1$. In this case for $\nu_0 = 33 \text{ GHz}$ we get $|g_{\phi\gamma}| < 5.13 \times 10^{-20} (G/B_{e0}) \text{ GeV}^{-1}$. After some algebraic operations in (6.17) and requiring that $P_C(T_0) < 7 \times 10^{-5}$ (MIPOL upper limit), we get the following constraint on $g_{\phi\gamma}$ from upper limit on the degree of circular polarization

$$|g_{\phi\gamma}| < 4.29 \times 10^{-19} (G/B_{e0}) \text{ GeV}^{-1}, \quad (\text{MIPOL}) \quad (6.18)$$

which is within the constraint (6.13) and we took $U_i \simeq -Q_i$ with $Q_i \simeq 10^{-6}$. The MIPOL upper limit (6.18) is also satisfied if $|g_{\phi\gamma}| \lesssim 5.13 \times 10^{-20} (G/B_{e0}) \text{ GeV}^{-1}$ for $211.38 \tilde{a}^{-1} b \lesssim 1$, which is a more conservative limit and satisfies both MIPOL upper limit and the constraint of perturbation theory.

In the domain of circular polarization, now it remains to study the last case when the term M_ϕ dominates the plasma term M_{pl} in $\Delta M_{1,2}$. Proceeding in the same way as above, we calculate the functions $\mathcal{G}_{1,2}$ which enter the trigonometric functions in I_γ and V . An important difference now is that we do not have to worry about the ionization fraction since it does not appear in M_ϕ . Inserting all necessary quantities into $\mathcal{G}_{1,2}$ we get

$$\mathcal{G}_1(T) \simeq \mathcal{G}_2(T) = 8 \times 10^{47} \left(\frac{m_\phi}{\text{eV}} \right)^2 \left(\frac{\text{Hz}}{\nu_0} \right) (T_i^{-5/2} - T^{-5/2}) \quad (\text{K}^{5/2}).$$

The next step is to calculate the double integrals \mathcal{I}_{cs} and \mathcal{I}_{sc} . However, in this case there are no known analytic solutions for both type of integrals so we must evaluate them numerically together with y for some specific values of the parameters. It would be more convenient first to write $y = b f_1(m_\phi, \nu_0)$ and $\mathcal{I}_{cs} - \mathcal{I}_{sc} = b f_2(m_\phi, \nu_0)$ and after calculate f_1 and f_2 numerically for given values of m_ϕ and ν_0 . Let us recall that now we are in the situation in which the constraints of perturbation theory are given by (6.12). We may consider for example $m_\phi = 10^{-8} \text{ eV}$ and $\nu_0 = 33 \text{ GHz}$ which corresponds to the working frequency of MIPOL experiment. Using numerical integration we obtain $f_1 = 9.37 \times 10^{-24}$ and $f_2 = 6.74 \times 10^{-11}$. Second, using the relation $P_C(T_0) = |-(\mathcal{G}(T_0) + b f_2) U_i| / (1 - y(1 - Q_i))$ we get the following constraint

$$|g_{\phi\gamma}| < 1.26 \times 10^{-16} (G/B_{e0}) \text{ GeV}^{-1}. \quad (6.19)$$

If we consider for example $m_\phi = 10^{-6} \text{ eV}$ for the same working frequency we would obtain $f_1 = 9.37 \times 10^{-32}$, $f_2 = 6.74 \times 10^{-15}$ and the above limit would be two orders of magnitude weaker.

6.3 No CMB polarization at decoupling

So far, in our treatment of generation of CMB polarization, we have assumed a priori that the CMB acquired a small polarization due to Thomson scattering at decoupling time, with non zero Stokes parameters Q_i and U_i . However, even though this assumption may seems reasonable, it is not accurate because of the fact that measurements of the CMB properties are done at present epoch and not at decoupling. In fact, if cosmological magnetic fields were present at decoupling epoch, production of pseudoscalar particles would generate CMB polarization independently on Thomson scattering and consequently there is no unambiguous way to tell, if the linear polarization experimentally observed at present is due to either Thomson scattering or photon-pseudoscalar particle mixing²⁵.

Another important assumption about generation of CMB polarization, is essentially generated at decoupling time and it remains invariant during subsequent evolution of the universe. Even this assumption is not completely end of the story, since for example the CMB may acquire a very small additional

²⁵Or due to another mechanism not considered in this work.

polarization during the reionization epoch due to Thomson scattering. So based on these two basic assumptions one would have that degree of linear polarization is generated by Thomson scattering and it remains almost invariant after the decoupling epoch.

In the previous sections we found upper limits/constraints on $g_{\phi\gamma}$ from current limit on the degree of circular polarization which is not directly generated by Thomson scattering. However, one may note that these upper limits/constraints found for $g_{\phi\gamma}$ in case of circular polarization, we have to high accuracy $P_L(T_0) \simeq P_L(T_i)$. These considerations would suggest that based on current limit on the degree of circular polarization and consequently on derived limits/constraints on $g_{\phi\gamma}$ from it, in principle the observed CMB linear polarization could be generated by a combination of Thomson scattering and magnetic-optic effects such as photon-pseudoscalar particle mixing where the latter is the subdominant component.

In this section, we consider another possible situation, the other way round, where the contribution of Thomson scattering to the linear polarization is supposed to be very small, so it can be completely neglected and generation of linear polarization is mostly due to photon-pseudoscalar particle mixing. In this case we would have that the CMB at decoupling is unpolarized with $Q_i = U_i = V_i = 0$. Let us concentrate for the moment on the degree of linear polarization which is given by $P_L(T) = (Q^2(T) + U^2(T))^{1/2}/I_\gamma(T)$. In the weak mixing case, from (5.12) and (5.9) and considering for example the case when $|M_{\text{pl}}| > |M_\phi|$, we get the following expression for $P_L(T)$ at present

$$P_L(T_0) = \frac{y}{1-y} \quad (\text{weak mixing and } |M_{\text{pl}}| > |M_\phi|), \quad (6.20)$$

where we assumed the CMB unpolarized at decoupling and $U(T) = 0$ for unpolarized light. Since P_L depends on $y = 4\tilde{a}^{-2}b(1 - \cos[52.84\tilde{a}])$, it explicitly depends on the photon frequency ν_0 . Even though one can easily calculate P_L at a given frequency, is more convenient to calculate its average value on a given interval. If we consider for example $10^8 \text{ Hz} \leq \nu_0 \leq 10^{11} \text{ Hz}$, the average value of $\tilde{a}^{-2}(1 - \cos[52.84\tilde{a}]) \simeq 2.35 \times 10^{-15}$. Considering that the degree of linear polarization of CMB at present is $P_L(T_0) \simeq 10^{-6}$, we get the following value for $\langle |g_{\phi\gamma}| \rangle \simeq 1.28 \times 10^{-18} (\text{G}/B_{e0})$.

In case of strong mixing, the degree of linear polarization for unpolarized CMB at decoupling is given by

$$P_L(T_0) = \frac{(1/2)\sin^2[F_{\phi\gamma}(T_0)]}{1 - (1/2)\sin^2[F_{\phi\gamma}(T_0)]}, \quad (6.21)$$

where we used expression (5.11). The solutions of trigonometric equation (6.21) in the strong mixing together with the constrain (6.1) (dictated by perturbation theory) are

$$g_{\phi\gamma}B_{e0} \simeq \{1.17 \times 10^{-24}(2n\pi \pm 0.0014), 1.17 \times 10^{-24}(2n\pi + 3.143), 1.17 \times 10^{-24}(2n\pi + 3.14)\},$$

where $n \gg 3.14 \times 10^{21}$ with $n \in \mathbf{Z}$ and took for simplicity $m_\phi = 10^{-8} \text{ eV}$ and $\nu_0 = 53 \text{ GHz}$ in (6.1). In the case when (6.2) applies, the solutions of (6.21) are given by

$$g_{\phi\gamma}B_{e0} \simeq \{1.17 \times 10^{-24}(2n\pi \pm 0.0014), 1.17 \times 10^{-24}(2n\pi + 3.143), 1.17 \times 10^{-24}(2n\pi + 3.14)\},$$

where $n \gg 8.1 \times 10^9$ with $n \in \mathbf{Z}$ and took for simplicity $\nu_0 = 53 \text{ GHz}$ in (6.2).

In the resonant case ($\Delta M_2(T) = 0$), as already mentioned in the previous sections, expressions for the Stokes parameters I_γ and Q are found exactly with the restrictions $g_{\phi\gamma} > 0$ and the mass of the pseudoscalar particle at present must be $m_\phi(T_0) \simeq 1.6 \times 10^{-14} \text{ eV}$. The expressions of Stokes parameters I_γ and Q in the resonant case coincide with those of strong mixing for $G_2(T) = 0$. Consequently, expression (6.21) is valid for both resonant and strong mixing cases for unpolarized CMB at decoupling. Therefore the solutions of (6.21) in the resonant case for $g_{\phi\gamma} > 0$ are

$$g_{\phi\gamma}B_{e0} \simeq 1.17 \times 10^{-24}(2n\pi - 0.0014) \quad \text{for } n \geq 1 \quad \text{or} \quad g_{\phi\gamma}B_{e0} \simeq \{1.17 \times 10^{-24}(2n\pi + 3.143), 1.17 \times 10^{-24}(2n\pi + 3.14), 1.17 \times 10^{-24}(2n\pi + 0.0014)\} \quad \text{for } n \geq 0. \quad (6.22)$$

We may note that in case when the argument of sine function in (6.21), $F_{\phi\gamma}(T_0)$ is less than unity (or $g_{\phi\gamma}B_{e0} < 1.17 \times 10^{-24}$) and because in general $P_L \ll 1$, from (6.21) one would get $\sin^2[F_{\phi\gamma}(T_0)] \simeq 2P_L(T_0)$. Considering that $P_L(T_0) \simeq 10^{-6}$, we get in the resonant case the following frequency independent value $|g_{\phi\gamma}| \simeq 1.66 \times 10^{-27}(\text{G}/B_{e0})$. This solution corresponds to the last set in (6.22) for $n = 0$. Another important thing to note in the case of linear polarization is $U(T) = 0$ for initially unpolarized CMB and consequently the angle of the polarization ellipse is $\tan[2\psi(T)] = 0$, which implies a horizontal linear polarization and no rotation of the polarization plane²⁶. In case of circular polarization, one may observe from (5.8) and (5.9) that $V(T) = 0$ for initially unpolarized CMB, which means no generation of circular polarization for transverse magnetic field.

Mixing of CMB photons with pseudoscalar particles would also generate CMB temperature anisotropy from an initially thermalized state²⁷. In order to prove this statement, consider the CMB at decoupling completely in thermal equilibrium and consequently unpolarized. Consider the evolution of intensity I_γ as a function of T for two specific observation directions: parallel and perpendicular to \mathbf{B}_e . In the direction parallel to \mathbf{B}_e , as we already have seen in Sec. 4, it is induced only the Faraday effect and other magneto-optic effects are absent. As we saw there, the intensity for parallel propagation changes only due universe expansion, $(T_i/T)^3 I_\gamma(T^\parallel) = I_\gamma(T_i^\parallel)$. The intensity observed parallel to \mathbf{B}_e is the same as that observed without the presence of the external field or unperturbed universe. On the other hand, the observed intensity for perpendicular propagation with respect to \mathbf{B}_e , for initially unpolarized CMB, is given by (5.11) which in case of resonant mixing is $(T_i/T)^3 I_\gamma(T^\perp) = I_\gamma(T_i^\perp) (1 - (1/2) \sin^2[F_{\phi\gamma}(T)])$. In case of thermalized CMB at decoupling we have that $I_\gamma(T_i^\perp) = I_\gamma(T_i^\parallel) = I_\gamma(T_i)$ where $I_\gamma(T_i)$ is the initial intensity at decoupling of the unperturbed black body photosphere, where for a black body $I_\gamma(T, \nu) = 4\pi\nu^2 [\exp(2\pi\nu/T) - 1]^{-1}$. Here T is the average value of the CMB over all directions.

To linear order in δT , we find the following relation from the black body intensity

$$\frac{\delta I_\gamma}{I_\gamma} = \left(\frac{x e^x}{e^x - 1} \right) \frac{\delta T}{T_0}, \quad (6.23)$$

where we defined $x \equiv 2\pi\nu_0/T_0$ and T_0 is the average value of the temperature at present, averaged over all directions in the sky. In case when $x < 1$ or $\nu_0 < 5.63 \times 10^{10}$ Hz, namely Rayleigh-Jeans regime, we have essentially $\delta I_\gamma/I_\gamma \simeq \delta T/T_0$, while for $x > 1$ (Wien regime) one must use the whole expression (6.23). We can use (6.23) to find the value of $g_{\phi\gamma}$ in the resonant case²⁸. Therefore, we have

$$\frac{I_\gamma(T_0^\parallel) - I_\gamma(T_0^\perp)}{I_\gamma(T_0^\parallel)} = \frac{1}{2} \sin^2[F_{\phi\gamma}(T_0)]. \quad (6.24)$$

The temperature anisotropy of the CMB depends on the angular separation between two points in the sky. Since we are comparing the intensity between parallel and perpendicular observations, this means an angular separation scale of 90° . According to WMPA9 collaboration [37], the temperature anisotropy at 90° or multipole moment $l \sim 1$, is $\delta T/T \simeq 3 \times 10^{-5}$. From (6.23) and (6.24) we get the following value of $g_{\phi\gamma}$

$$|g_{\phi\gamma}| \simeq 9.12 \times 10^{-27} \left(\frac{x e^x}{e^x - 1} \right)^{1/2} (\text{G}/B_{e0}) \text{ GeV}^{-1},$$

where we considered for simplicity only the case when $F_{\phi\gamma}(T_0) < 1$ and assumed that the contribution of large scale magnetic field to temperature anisotropy is subdominant to photon-pseudoscalar

²⁶This conclusion applies for unpolarized CMB at decoupling and for transverse magnetic field.

²⁷In addition to generation of CMB temperature anisotropy by photon-pseudoscalar mixing, there is also generation of temperature anisotropy by the large scale magnetic field itself. Consequently the total temperature anisotropy is given by the sum of photon-pseudoscalar mixing and magnetic field temperature anisotropy contributions.

²⁸Here we consider for simplicity only the resonant case, however expression (6.24) is also valid in the strong mixing case. In general given the value of temperature anisotropy, the trigonometric Eq. (6.24) has multiple solutions in both strong and resonant mixing regimes.

mixing contribution. In general (6.24) has multiple solutions similar to (6.22) when $F(T_0) > 1$. In the Rayleigh-Jeans part of the spectrum we get $|g_{\phi\gamma}| \simeq 9.12 \times 10^{-27} (G/B_{e0}) \text{ GeV}^{-1}$.

In the weak mixing case and for $|M_{\text{pl}}| > |M_\phi|$ we get

$$\frac{I_\gamma(T_0^\parallel) - I_\gamma(T_0^\perp)}{I_\gamma(T_0^\parallel)} = y = 4 \tilde{a}^{-2} b(1 - \cos[52.84 \tilde{a}]), \quad (6.25)$$

where we used the expression for I_γ in (5.12) with $Q_i = 0$ and used the definition²⁹ y for $|M_{\text{pl}}| > |M_\phi|$. As we can see from (6.25), δI_γ is proportional to the fast varying function, $1 - \cos[52.84 \tilde{a}]$, which for $\tilde{a} = 2n\pi/52.84$ is equal to zero, with $n \in \mathbf{Z}$, independently on the value of $g_{\phi\gamma}$. In case when $|M_{\text{pl}}| < |M_\phi|$, the value of $y = bf_1(m_\phi, \nu_0)$ can be calculated numerically as we did for the case of circular polarization for given values of m_ϕ and ν_0 .

7 Discussion and conclusions

In this work we have studied the most important magneto-optic effects and their impact in the generation of CMB polarization. We presented a systematic study of each of them where we mostly focused on the generation of CMB circular polarization. In this work we found the equations of motion for photon and pseudoscalar fields in an external magnetic field in the WKB approximation, and then found the equations of motion for the Stokes parameters by using density matrix approach as shown in Sec. 3. The resulting equations describe the mixing of different magneto-optic effects which obviously complicate the situation but on the other hand give richer scenarios.

In Sec. 4 we studied the vacuum polarization and CM effects separately, in order to isolate the contribution of each of them to CMB polarization. They are second order magneto-optic effects on magnetic field amplitude B_e and are responsible for generation of phase shifts between the states A_+ and A_\times . These effects generate CMB elliptical polarization only in the case when the CMB is initially polarized. In this work we concentrated in the post decoupling epoch and worked under the hypothesis that the CMB acquired a small polarization at decoupling time due to Thomson scattering. We used perturbation theory and found the evolution as a function of T of the Stokes parameters. We studied in particular the generation of circular polarization which is represented by the Stokes parameter V , in cases of observation angles $\Phi \neq \pi/2$ and $\Phi = \pi/2$.

The contribution of vacuum polarization and CM effects to V depends essentially on Φ , B_{e0} , ν_0 and on the magnitude of the Stokes parameters at decoupling which, on the other hand, depend on the temperature anisotropy. In this work we assumed that $V_i = 0$ at decoupling while the other parameters are non zero. The magnitude of the parameters Q_i and U_i obviously are smaller than temperature anisotropy and observations of CMB linear polarization by DASI, WMAP and BOOMERANG give an order of magnitude of $Q_i \sim U_i \sim 10^{-6}$.

In the case of vacuum polarization and $\Phi \neq \pi/2$, the degree of circular polarization is proportional to Q_i and U_i , as shown in (4.20) and in most cases is the term proportional to U_i which dominates. This term on the other hand is proportional to ν_0 and B_{e0}^2 . Consequently, significant generation of circular polarization would occur in the high frequency part of the CMB and for higher values of B_{e0} . In this work we used in our estimates a canonical value of $B_{e0} \sim \text{nG}$ but in principle higher values are possible. If for example one observes the CMB in the Wien region, say at $\nu_0 \simeq 700 \text{ GHz}$ and the magnetic field is of the order of 100 nG , the degree of circular polarization would be of the order $P_C \sim 10^{-11}$ while for $B_{e0} \sim \text{nG}$ is four orders of magnitude smaller.

Also for the CM effect, the degree of circular polarization is proportional to the initial values of Stokes parameters at decoupling and to B_{e0}, ν_0 and Φ . One distinguishing feature of the CM effect is the relation between V_0 and ν_0 which is $V_0 \propto \nu_0^{-3}$. This relation makes the CM effect quite appealing in regard to generation of circular polarization in the Rayleigh-Jeans part of the spectrum. For $\Phi \neq \pi/2$ the

²⁹ y should not be confused with the Compton y -parameter used in the CMB spectral distortion.

degree of circular polarization is given in (4.33) where the first term is proportional to Q_i and the second term is proportional to U_i . The term proportional to Q_i shares a common feature with the vacuum polarization by the fact it gets contribution from the Faraday effect which is encoded in ρ . Under the approximations used in Sec. 4, the term proportional to Q_i is smaller than that proportional to U_i . The latter coincides with the solution found for V in case when $\Phi = \pi/2$ for $\mathcal{G}(T) \ll 1$ and $F(T) < 1$. This means that the contribution of the CM effect to circular polarization is bigger in the limit $\Phi \rightarrow \pi/2$. The degree of circular polarization is substantive in the frequency region $\nu_0 \sim 10^8 - 10^9$ Hz while for higher frequencies $\nu_0 \sim 10^{11}$ Hz, the contribution of CM effect to V_0 is subdominant to vacuum polarization. For example, if $\nu_0 \sim 10^8$ Hz and $B_{e0} \sim$ nG, the degree of circular polarization for the CM effect would be $P_C \sim 10^{-10}$ while if $B_{e0} \sim 100$ nG, $P_C \sim 10^{-6}$.

In this work, we also studied the generation of elliptic polarization due to photon-pseudoscalar particle mixing in cosmic magnetic field, with emphasis on the degree of circular polarization. Differently from the vacuum polarization and CM effects, photon-pseudoscalar mixing has in addition two more independent parameters which are m_ϕ and $g_{\phi\gamma}$. We studied this mechanism in case of only transverse magnetic field and used perturbation theory to find the evolution in T of the Stokes parameters. We used perturbation theory in two mixing regimes, namely weak and strong mixing and estimated the degree of circular polarization at present epoch.

Since the parameters $g_{\phi\gamma}$ and m_ϕ are free and in general span a wide range of values, we used the present upper limit on the degree of circular polarization in order to constrain $g_{\phi\gamma}$ and m_ϕ . These parameters on the other hand are constrained by the mixing regimes, therefore the limits that we presented are valid in these regimes. In the strong mixing regime, in general one has to solve trigonometric equations or inequations which have as independent variable $g_{\phi\gamma}B_{e0}$. The solutions generally, depend on an integer number n and consequently they are not unique. The interval of values of $g_{\phi\gamma}B_{e0}$ can in principle be narrowed by complementary constraints on $g_{\phi\gamma}$ from other methods. On the other hand, in the weak mixing case there is not such a dependence on n . In this case by using the upper limit on P_C obtained from MIPOL experiment, we got the constraint $|g_{\phi\gamma}| < 4.29 \times 10^{-19}(\text{G}/B_{e0}) \text{ GeV}^{-1}$ for $m_\phi < 1.6 \times 10^{-14} \text{ eV}$.

Other limits have been obtained from the degree of linear polarization and by considering the case of unpolarized CMB at decoupling. In the weak mixing case, we obtained the average value over frequency of $\langle |g_{\phi\gamma}| \rangle \sim 10^{-18}(\text{G}/B_{e0})$ for $m_\phi < 1.6 \times 10^{-14} \text{ eV}$ and $P_L(T_0) \simeq 10^{-6}$. In the strong mixing case, again one obtains values of $g_{\phi\gamma}B_{e0}$ that depends on n and therefore there is no unique solution. The same thing happens even in the resonant case with the particular case that, if, $g_{\phi\gamma}B_{e0} < 1.17 \times 10^{-24}$ (obtained by other methods), then from $P_L \simeq 10^{-6}$ we get the value $|g_{\phi\gamma}| \simeq 1.66 \times 10^{-27}(\text{G}/B_{e0})$ for $m_\phi \simeq 1.6 \times 10^{-14} \text{ eV}$. As in the case of vacuum polarization and CM effects, photon-pseudoscalar particle mixing generates non uniform degree of circular polarization and rotation of the polarization plane across the sky. This fact can be used in order to understand if the observed linear polarization at present has a non uniform component across the sky. If this would be true, it might be due to photon-pseudoscalar particle mixing if it is the dominant mechanism of generation of linear polarization among magneto-optic effects studied.

From the experimental side, it turns out that among CM and QED effects, the CM effect is the most promising effect on generating circular polarization in the low frequency part of the CMB due to the dependence $V_0 \propto \nu_0^{-3}$, while the vacuum polarization is the dominant one in the high frequency part due to $V_0 \propto \nu_0$. The degree of circular polarization due to the CM effect in the low frequency part, in general, is bigger than that generated by vacuum polarization at high frequencies. Moreover, vacuum polarization and CM effects generate a rotation of the polarization plane of the CMB and this rotation together with the degree of circular polarization are not uniform across the sky because they depend on the observation angle Φ . These facts would suggest that observation of CMB circular polarization is more likely to happen in the low frequency part of the CMB, mostly due to the CM effect and if, the observation frequency range is not a big detection issue. On the other hand, if one is interested in the measurement of the rotation angle of the polarization plane, the non uniformity of the rotation across the sky might be an issue.

In order to detect CMB circular polarization, probably the most convenient frequency range would be for $\nu_0 \sim 10^8 - 10^9$ Hz where the degree of circular polarization would be in the interval $P_C \simeq 10^{-13} - 10^{-10}$ for $B_{e0} \simeq 1$ nG due to CM effect where the higher value of P_C corresponds to the lower value of ν_0 . In this frequency range the contribution of vacuum polarization is completely negligible with respect to CM effect. The vacuum polarization is dominant to the CM effect in the Wien regime and for $\nu_0 \sim 700$ GHz we found the interval $P_C \lesssim 10^{-15} - 10^{-11}$ for the interval $B_{e0} \lesssim 1 - 100$ nG. Assuming for the moment that frequency observation range is not an issue and it can be fixed based on experiment characteristic, one main problem on the detection of CMB circular polarization is related to the magnetic field amplitude which is poorly known. In the case of intergalactic magnetic field, usually upper limits are found by different methods and its amplitude is expected to be less than 1 nG (canonical value) up to a value of less than 100 nG. Using a canonical value of $B_{e0} \lesssim 1$ nG, one would expect that the degree of circular polarization to be $P_C \lesssim 10^{-10}$ in the Rayleigh-Jeans regime and $P_C \lesssim 10^{-15}$ in the Wien region. These values of the degree of circular polarization corresponds essentially to the case when $\Phi = \pi/2$ and for different values of Φ , P_C is usually smaller and not uniform across the sky.

The photon-pseudoscalar mixing contributes to circular polarization as well and based on values of $g_{\phi\gamma}$ and m_ϕ found from current upper limit on circular polarization from MIPOL experiment, its contribution might be bigger than CM and vacuum polarization effects. So, let us assume for example that $B_{e0} \lesssim 1$ nG and circular polarization would be detected with degree of circular polarization with value in the range, $10^{-10} \lesssim P_C \lesssim 10^{-6}$. This would mean there is a contribution to P_C due to photon-pseudoscalar mixing which is much bigger than other magneto-optic effects or the amplitude of the magnetic field might be higher than assumed or circular polarization is generated by another effect not considered in this work. If one would detect CMB circular polarization with average value of $P_C \lesssim 10^{-10}$, it is more likely that CM and vacuum polarization effects are the source of this polarization.

It is worth to mention also what has not been studied in this work. The first thing is related to Thomson scattering and scattering of pseudoscalar particles at post decoupling epoch, namely for $T < 2970$ K and their absence in our density matrix formalism. In general, scattering is a mechanism of coherence breaking for mixing/oscillation processes which results in damping of the fields. In the density matrix formalism, the structure of the damping operator can be calculated by using field theory for scattering which is essentially the calculation of the commutator $[H_T, \rho]$ on the r. h. s. of (3.3) where H_T includes the Hamiltonian for the Thomson scattering and that of scattering of pseudoscalar particles. However, quite often the damping term due to scattering, in case of non degenerate and non relativistic electron gas can be approximated³⁰ by, $-i\{\Gamma, \rho\}$, where Γ is the scattering rate matrix of photons and pseudoscalar particles which is diagonal in the basis $|A_+\rangle, |A_\times\rangle, |\phi\rangle$. Consequently, the damping term due to scattering, would have the same structure as the damping term due to Hubble friction. Therefore, the Stokes parameters would be affected by scattering but not their ratio because it cancels out exactly as the damping term due to Hubble friction. However, one must always keep in mind that this is an crude approximation.

The second thing is related to the case $\Phi \neq \pi/2$ for the photon-pseudoscalar particle mixing. In Sec. 5, we found the equations of motion for the reduced Stokes vectors in case of transverse external magnetic field. In this case it was possible to find two sets of decoupled differential equations for the reduced Stokes vectors and solved the equations by using perturbation theory. If the field is not transverse, namely $\Phi \neq \pi/2$, in general one has to solve simultaneously, a system of nine linear differential equations of the first order which can be problematic to solve even numerically because quite often they are stiff. We shall treat this problem in more details elsewhere but even at this stage we can outline very important conclusions about the nature of the solutions and the impact on the CMB polarization.

The importance of solutions of the equations of motion in the case $\Phi \neq \pi/2$ (for photon-pseudoscalar mixing) relies in the fact, that being the system of equations linear, see Eqs. (5.5), the solutions will be proportional to initial values at a given temperature T_i which does necessarily coincides with decoupling temperature. Therefore, each Stokes parameter would be proportional to $I_\gamma(T_i), Q_i, U_i, V_i, S_{4i}$ etc., and

³⁰Similar situation occurs quite often in neutrino physics, see Ref. [38].

for $T < T_i$ usual Stokes parameters (those which in general interest us) would be different from zero even in case of initially unpolarized CMB at $T = T_i$. We saw similar situation in Sec. 6.3, where we studied the case of unpolarized CMB at decoupling for transverse magnetic field. Consequently, the CMB would acquire elliptic polarization independently on Thomson scattering, even in case when it is initially unpolarized.

This situation would be very important in order to investigate prior decoupling CMB polarization due to photon-pseudoscalar mixing in external magnetic field. According to standard cosmology, generation of CMB polarization occurs at or very close to decoupling time due to Thomson scattering when the condition of tight coupling between photons and electro-baryon plasma breaks down. Indeed, for most models of generation of CMB polarization which include scalar perturbations, magnetic fields, gravitational waves etc., at the end is always the Thomson scattering which generates CMB polarization [39]. The tight coupling condition would imply that, if there is any degree of polarization prior to decoupling, generated at temperature T , it would be damped very fast due to scattering of photons with electrons and baryons. However, as we have seen and discussed in this work, photon-pseudoscalar mixing apart from generating temperature anisotropy as shown in Sec. 6.3 and spectral distortions of the CMB [40], it generates also polarization, independently on Thomson scattering. Consequently, here we advance the hypothesis that photon-pseudoscalar mixing *might* generate non uniform CMB polarization across the sky, even before decoupling epoch, if the rate of photon-pseudoscalar oscillation is faster than photon scattering rate with electro-baryon plasma. Obviously, all said about this hypothesis would depend on pseudoscalar field parameters m_ϕ and $g_{\phi\gamma}$. The suggested hypothesis needs further attentive study and it would be too premature to conclude that it is indeed the case.

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A Appendix

In Sec. 5 we have derived expressions for usual Stokes parameters I_γ, Q, U and V by using perturbation theory for weak and strong mixing cases in transverse external magnetic field. In this appendix we show for completeness reasons the solutions of remaining Stokes parameters, namely S_4, S_5, S_6, S_7 and S_8 for weak and strong mixing cases. The parameters S_4 and S_5 are components of the first reduced Stokes vector \tilde{S}_1 while S_6, S_7 and S_8 are components of the second reduced Stokes vector \tilde{S}_2 .

In case of strong mixing $|\Delta M| < |\Delta M_1| \ll M_{\phi\gamma}$ or $|\Delta M_1| < |\Delta M| \ll M_{\phi\gamma}$, solutions of the Stokes parameters S_4 and S_5 to first order in perturbation theory, are respectively given by

$$\begin{aligned} \left(\frac{T_i}{T}\right)^3 S_4(T) = & \sin[F_{\phi\gamma}(T)] U_i + \left[\sin[F_{\phi\gamma}(T)] \int_T^{T_i} (G(T') \cos^2[F_{\phi\gamma}(T')] + G_1(T') \sin^2[F_{\phi\gamma}(T')]) dT' - \frac{1}{2} \cos[F_{\phi\gamma}(T)] \right. \\ & \times \int_T^{T_i} \Delta G(T') \sin[2F_{\phi\gamma}(T')] dT' \Big] V_i + \cos[F_{\phi\gamma}(T)] S_{4i} + \left(\cos[F_{\phi\gamma}(T)] \int_T^{T_i} (G(T') \sin^2[F_{\phi\gamma}(T')] \right. \\ & \left. \left. + G_1(T') \cos^2[F_{\phi\gamma}(T')]) dT' - \frac{1}{2} \sin[F_{\phi\gamma}(T)] \int_T^{T_i} \Delta G(T') \sin[2F_{\phi\gamma}(T')] dT' \right) S_{5i}, \end{aligned}$$

$$\begin{aligned}
\left(\frac{T_i}{T}\right)^3 S_5(T) = & \left(-\sin[F_{\phi\gamma}(T)] \int_T^{T_i} (G(T') \cos^2[F_{\phi\gamma}(T')] + G_1(T') \sin^2[F_{\phi\gamma}(T')]) dT' + \frac{1}{2} \cos[F_{\phi\gamma}(T)] \right. \\
& \times \int_T^{T_i} \Delta G(T') \sin[2F_{\phi\gamma}(T')] dT' \Big) U_i + \sin[F_{\phi\gamma}(T)] V_i - \left(\cos[F_{\phi\gamma}(T)] \int_T^{T_i} (G(T') \sin^2[F_{\phi\gamma}(T')] \right. \\
& \left. + G_1(T') \cos^2[F_{\phi\gamma}(T')]) dT' - \frac{1}{2} \sin[F_{\phi\gamma}(T)] \int_T^{T_i} \Delta G(T') \sin[2F_{\phi\gamma}(T')] dT' \right) S_{4i} + \cos[F_{\phi\gamma}(T)] S_{5i}.
\end{aligned} \tag{A.1}$$

Now for the opposite case or weak mixing regime, in accordance with Sec. 5 we have $|\Delta M| < M_{\phi\gamma} \ll |\Delta M_1|$. In Sec. 5 we found the solutions for U and V up to second order in ϵ in perturbation theory and consequently the expressions for S_4 and S_5 are respectively given by

$$\begin{aligned}
\left(\frac{T_i}{T}\right)^3 S_4(T) = & \left[\cos[\mathcal{G}_1(T)] \int_T^{T_i} G_{\phi\gamma}(T') \cos[\mathcal{G}_1(T')] dT' + \cos[\mathcal{G}_1(T)] \int_T^{T_i} G_{\phi\gamma}(T') \sin[\mathcal{G}_1(T')] dT' \int_{T'}^{T_i} G(T'') dT'' \right. \\
& + \sin[\mathcal{G}_1(T)] \int_T^{T_i} G_{\phi\gamma}(T') \sin[\mathcal{G}_1(T')] dT' - \sin[\mathcal{G}_1(T)] \int_T^{T_i} G_{\phi\gamma}(T') \cos[\mathcal{G}_1(T')] dT' \int_{T'}^{T_i} G(T'') dT'' \Big] U_i \\
& + \left[-\cos[\mathcal{G}_1(T)] \int_T^{T_i} G_{\phi\gamma}(T') \sin[\mathcal{G}_1(T')] dT' + \cos[\mathcal{G}_1(T)] \int_T^{T_i} G_{\phi\gamma}(T') \cos[\mathcal{G}_1(T')] dT' \int_{T'}^{T_i} G(T'') dT'' \right. \\
& + \sin[\mathcal{G}_1(T)] \int_T^{T_i} G_{\phi\gamma}(T') \cos[\mathcal{G}_1(T')] dT' + \sin[\mathcal{G}_1(T)] \int_T^{T_i} G_{\phi\gamma}(T') \sin[\mathcal{G}_1(T')] dT' \int_{T'}^{T_i} G(T'') dT'' \Big] V_i \\
& + \left[\cos[\mathcal{G}_1(T)] \left(\int_T^{T_i} G_{\phi\gamma}(T') \left(\cos[\mathcal{G}_1(T')] \int_{T'}^{T_i} G_{\phi\gamma}(T'') (-\cos[\mathcal{G}_1(T'')]) dT'' - \sin[\mathcal{G}_1(T')] \right. \right. \right. \\
& \times \left. \left. \left(\int_{T'}^{T_i} G_{\phi\gamma}(T'') \sin[\mathcal{G}_1(T'')] dT'' \right) \right) dT' + \sin[\mathcal{G}_1(T)] \left(\int_T^{T_i} G_{\phi\gamma}(T') (\cos[\mathcal{G}_1(T')] \times \right. \right. \\
& \left. \left. \int_{T'}^{T_i} G_{\phi\gamma}(T'') \sin[\mathcal{G}_1(T'')] dT'' + \sin[\mathcal{G}_1(T')] \left(\int_{T'}^{T_i} G_{\phi\gamma}(T'') (-\cos[\mathcal{G}_1(T'')]) dT'' \right) \right) dT' \right) \\
& + \cos[\mathcal{G}_1(T)] S_{4i} + \left[\cos[\mathcal{G}_1(T)] \int_T^{T_i} G_{\phi\gamma}(T') \left(\cos[\mathcal{G}_1(T')] \int_{T'}^{T_i} G_{\phi\gamma}(T'') (-\sin[\mathcal{G}_1(T'')]) dT'' \right. \right. \\
& - \sin[\mathcal{G}_1(T')] \left(\int_{T'}^{T_i} G_{\phi\gamma}(T'') (-\cos[\mathcal{G}_1(T'')]) dT'' \right) \Big) dT' + \sin[\mathcal{G}_1(T)] \left(\int_T^{T_i} G_{\phi\gamma}(T') \right. \\
& \times \left(\sin[\mathcal{G}_1(T')] \left(\int_{T'}^{T_i} G_{\phi\gamma}(T'') (-\sin[\mathcal{G}_1(T'')]) dT'' \right) + \cos[\mathcal{G}_1(T')] \int_{T'}^{T_i} G_{\phi\gamma}(T'') \right. \\
& \left. \left. \times (-\cos[\mathcal{G}_1(T'')]) dT'' + 1 \right) \right] S_{5i}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{T_i}{T}\right)^3 S_5(T) = & \left[\cos[\mathcal{G}_1(T)] \int_T^{T_i} G_{\phi\gamma}(T') \cos[\mathcal{G}_1(T')] \left(\int_{T'}^{T_i} -G(T'') dT'' \right) dT' + \cos[\mathcal{G}_1(T)] \times \right. \\
& \int_T^{T_i} G_{\phi\gamma}(T') \sin[\mathcal{G}_1(T')] dT' + \sin[\mathcal{G}_1(T)] \left(\int_T^{T_i} G_{\phi\gamma}(T') \sin[\mathcal{G}_1(T')] \left(\int_{T'}^{T_i} -G(T'') dT'' \right) dT' - \right. \\
& \left. \left. \int_T^{T_i} G_{\phi\gamma}(T') \cos[\mathcal{G}_1(T')] dT' \right) \right] U_i + \left[\cos[\mathcal{G}_1(T)] \int_T^{T_i} G_{\phi\gamma}(T') \cos[\mathcal{G}_1(T')] dT' + \cos[\mathcal{G}_1(T)] \times \right. \\
& \int_T^{T_i} G_{\phi\gamma}(T') \sin[\mathcal{G}_1(T')] \left(\int_{T'}^{T_i} \mathcal{G}(T'') dT'' \right) dT' - \sin[\mathcal{G}_1(T)] \left(\int_T^{T_i} G_{\phi\gamma}(T') (-\sin[\mathcal{G}_1(T)]) dT' + \right. \\
& \left. \int_T^{T_i} G_{\phi\gamma}(T') \cos[\mathcal{G}_1(T)] \left(\int_{T'}^{T_i} \mathcal{G}(T'') dT'' \right) dT' \right] V_i + \left[\cos[\mathcal{G}_1(T)] \int_T^{T_i} G_{\phi\gamma}(T') (\cos[\mathcal{G}_1(T')]) \times \right. \\
& \left. \int_{T'}^{T_i} G_{\phi\gamma}(T'') \sin[\mathcal{G}_1(T'')] dT'' + \sin[\mathcal{G}_1(T')] \left(\int_{T'}^{T_i} G_{\phi\gamma}(T'') (-\cos[\mathcal{G}_1(T'')]) dT'' \right) \right] dT' - \\
& \sin[\mathcal{G}_1(T)] \left(\int_T^{T_i} G_{\phi\gamma}(T') \left(\cos[\mathcal{G}_1(T')] \int_{T'}^{T_i} G_{\phi\gamma}(T'') (-\cos[\mathcal{G}_1(T'')]) dT'' - \sin[\mathcal{G}_1(T')] \times \right. \right. \\
& \left. \left. \left(\int_{T'}^{T_i} G_{\phi\gamma}(T'') \sin[\mathcal{G}_1(T'')] dT'' \right) \right) dT' + 1 \right] S_{4i} + \left[\cos[\mathcal{G}_1(T)] \left(\int_T^{T_i} G_{\phi\gamma}(T') (\sin[\mathcal{G}_1(T')]) \times \right. \right. \\
& \left. \left. \left(\int_{T'}^{T_i} G_{\phi\gamma}(T'') (-\sin[\mathcal{G}_1(T'')]) dT'' \right) + \cos[\mathcal{G}_1(T')] \int_{T'}^{T_i} G_{\phi\gamma}(T'') (-\cos[\mathcal{G}_1(T'')]) dT'' \right) dT' \right] - \\
& \sin[\mathcal{G}_1(T)] \left(\int_T^{T_i} G_{\phi\gamma}(T') \left(\cos[\mathcal{G}_1(T')] \int_{T'}^{T_i} G_{\phi\gamma}(T'') (-\sin[\mathcal{G}_1(T'')]) dT'' - \sin[\mathcal{G}_1(T')] \times \right. \right. \\
& \left. \left. \left(\int_{T'}^{T_i} G_{\phi\gamma}(T'') (-\cos[\mathcal{G}_1(T'')]) dT'' \right) \right) dT' + \cos[\mathcal{G}_1(T)] \right] S_{5i}. \tag{A.2}
\end{aligned}$$

Now let us concentrate on the second reduced Stokes vector \tilde{S}_2 and on the components S_6, S_7, S_8 . Following Sec. 5, in the case when $|\Delta M_2| \ll M_{\phi\gamma}$, the solutions for the components S_6, S_7, S_8 up to the first order in the parameter ϵ are given by:

$$\begin{aligned}
\left(\frac{T_i}{T}\right)^3 S_6(T) = & -\frac{1}{2} \sin[2F_{\phi\gamma}(T)] Q_i + \cos[2F_{\phi\gamma}(T)] S_{6i} + \left[\sin[2F_{\phi\gamma}(T)] \int_T^{T_i} G_2(T') \sin[2F_{\phi\gamma}(T')] dT' \right. \\
& \left. + \cos[2F_{\phi\gamma}(T)] \int_T^{T_i} G_2(T') \cos[2F_{\phi\gamma}(T')] dT' \right] S_{7i} + \frac{\sqrt{3}}{2} \sin[2F_{\phi\gamma}(T)] S_{8i}, \\
\left(\frac{T_i}{T}\right)^3 S_7(T) = & \frac{1}{2} \left[\int_T^{T_i} G_2(T') \sin[2F_{\phi\gamma}(T')] dT' \right] Q_i - \left[\int_T^{T_i} G_2(T') \cos[2F_{\phi\gamma}(T')] dT' \right] S_{6i} + S_{7i} \\
& - \frac{\sqrt{3}}{2} \left[\int_T^{T_i} G_2(T') \sin[2F_{\phi\gamma}(T')] dT' \right] S_{8i}, \\
\left(\frac{T_i}{T}\right)^3 S_8(T) = & \frac{\sqrt{3}}{2} \sin^2[F_{\phi\gamma}(T)] Q_i - \frac{\sqrt{3}}{2} \sin[2F_{\phi\gamma}(T)] S_{6i} + \left[\frac{\sqrt{3}}{2} \cos[2F_{\phi\gamma}(T)] \int_T^{T_i} G_2(T') \sin[2F_{\phi\gamma}(T')] dT' \right. \\
& \left. - \frac{\sqrt{3}}{2} \sin[2F_{\phi\gamma}(T)] \int_T^{T_i} G_2(T') \cos[2F_{\phi\gamma}(T')] dT' \right] S_{7i} + \frac{1}{4} (1 + 3 \cos[2F_{\phi\gamma}(T)]) S_{8i}, \tag{A.3}
\end{aligned}$$

For the opposite case, namely for $|\Delta M_2| \gg M_{\phi\gamma}$, again following Sec. 5, the solutions for the

components S_6, S_7, S_8 up to the second order in the parameter ϵ are given by

$$\begin{aligned}
\left(\frac{T_i}{T}\right)^3 S_6(T) = & - \left[\cos[\mathcal{G}_2(T)] \int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' + \sin[\mathcal{G}_2(T)] \int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \right] Q_i \\
& \left[\cos[\mathcal{G}_2(T)] - 4 \cos[\mathcal{G}_2(T)] \int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \cos[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' - \right. \\
& 4 \sin[\mathcal{G}_2(T)] \int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \cos[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' \left. \right] S_{6i} - \\
& \left[4 \cos[\mathcal{G}_2(T)] \int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \sin[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' - \sin[\mathcal{G}_2(T)] (1 \right. \\
& \left. - 4 \int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \sin[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' \right) \left. \right] S_{7i} \\
& + \sqrt{3} \left[\cos[\mathcal{G}_2(T)] \int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' + \sin[\mathcal{G}_2(T)] \int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \right] S_{8i}, \\
\left(\frac{T_i}{T}\right)^3 S_7(T) = & \left[-\cos[\mathcal{G}_2(T)] \int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' + \sin[\mathcal{G}_2(T)] \int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \right] Q_i \\
& \left[-4 \cos[\mathcal{G}_2(T)] \int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \cos[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' + \sin[\mathcal{G}_2(T)] (-1 \right. \\
& \left. + 4 \int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \cos[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' \right) \left. \right] S_{6i} + \\
& \left[\cos[\mathcal{G}_2(T)] - 4 \cos[\mathcal{G}_2(T)] \int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \sin[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' + \right. \\
& 4 \sin[\mathcal{G}_2(T)] \int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \sin[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' \left. \right] S_{7i} \\
& + \sqrt{3} \left[\cos[\mathcal{G}_2(T)] \int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' - \sin[\mathcal{G}_2(T)] \int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \right] S_{8i}, \\
\left(\frac{T_i}{T}\right)^3 S_8(T) = & \sqrt{3} \left[\int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \cos[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' + \int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \times \right. \\
& \left. \int_{T'}^{T_i} \sin[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' \right] Q_i - \sqrt{3} \left[\int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \right] S_{6i} - \\
& \sqrt{3} \left[\int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \right] S_{7i} + \left[1 - 3 \int_T^{T_i} \cos[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \cos[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' \right. \\
& \left. - 3 \int_T^{T_i} \sin[\mathcal{G}_2(T')] G_{\phi\gamma}(T') dT' \int_{T'}^{T_i} \sin[\mathcal{G}_2(T'')] G_{\phi\gamma}(T'') dT'' \right] S_{8i}. \tag{A.4}
\end{aligned}$$

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